# THE CONSTRUCTION OF INDEX NUMBERS

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# THE CONSTRUCTION OF INDEX NUMBERS

BY

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Gift
Prof. Frederick E. Croxton

# PREFACE

In the academic year 1927–28 I gave the graduate course in Statistical Theory and Analysis at Harvard University. The point of view of statistics which I emphasized in that course was that the subject is not a collection of numerical facts, but rather a method and a logic; that the science of statistics is an applied science in which it is just as necessary to scrutinize the premises as the logic and mathematics; that mechanistic and universalistic notions of statistics are mistaken; that statistical methods are frequently dependent upon the peculiar data to which they are applied; that such methods are sound when they can be retained in actual practice, even though the methods are not applicable to data of another sort; and that economic statistics has the same relation to economics and mathematics as mathematical physics has to physics and mathematics. The construction of index numbers was discussed from the point of view just indicated, that is, as a problem necessarily involving not only economics and mathematics, but also the nature of the data utilized. The discussion referred to is presented in the following pages.

Warren M. Persons

*August* 15, 1928.

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# THE CONSTRUCTION OF INDEX NUMBERS

IF the price ($p_1$) of any commodity, such as wheat, is \$1.25 per bushel in period 1 and the price ($p_2$) in period 2 is \$1.50 per bushel, the relative price in period 2 as compared with period 1 ($p_2/p_1$) is $\$1.50 \div \$1.25 = 1.20 = 120$ per cent. The relative price in period 1 as compared with period 2 ($p_1/p_2$) is $\$1.25 \div \$1.50 = 0.83\frac{1}{3} =$ $83\frac{1}{3}$ per cent. The product of the two *price relatives* computed forward and backward ($p_2/p_1 \cdot p_1/p_2$), obviously, always equals 1.

If the prices ($p'_1$ and $p'_2$) of a second commodity, such as pig iron, are \$25 per ton and \$20 per ton in periods 1 and 2, respectively, the two price relatives, computed forward and backward, will be 80 per cent and 125 per cent.

If our object be the comparison of the *actual price change* of wheat per unit of quantity with that of pig iron per unit of quantity, between two periods, the units of quantity which we use for the two commodities, whether pounds, bushels or tons, will, of course, determine the size of the actual increase or decrease. That is, the units of quantity selected determine the magnitude of the increments of prices of the two commodities.

If we are interested in the comparison of the *relative price change* of wheat with that of pig iron, during the interval between periods 1 and

2, and take the ratios of actual prices instead of the differences between them, the units of quantity used for the quotation of prices of the two commodities do not affect the result. That is, the price relatives for two or more commodities are independent of the units of quantity used for the several commodities and, being expressed in a common unit (percentage), are *prima facie* comparable.

The object of the present study is that of measuring the average change of prices of two or more commodities during two or more periods. In other words, our problem is that of averaging a number of price relatives.

Starting with the basic concept of the relative price change of a single commodity during one interval, we then compute price relatives for a number of commodities, intervals, and bases and observe that in actual practice whatever arbitrary period we use as the base, we obtain magnitudes differing substantially from each other for the several commodities. The results for special cases indicate two empirical generalizations:

(*a*) The magnitude of the price relatives depends upon the base which we arbitrarily select.

(*b*) The distribution of the price relatives with respect to each other depends upon the base which we arbitrarily select.

These generalizations at once suggest a question. If we *assume an arbitrary base*, is there a plane or level of distribution of price relatives — constant for a given period — which is superior to other magnitudes as a concept to be retained in actual practice? This is the central problem of averages.

"Averaging" magnitudes is associated with familiar mathematical concepts and processes and, in order to obtain an average or summary figure which is a useful working concept, we must utilize these familiar mathematical processes. The process may be that of summation, leading to the arithmetic and harmonic means, or of multiplication, leading to the geometric mean, or of arranging the items in order of magnitude, leading to the median, or of finding the item which appears most frequently, leading to the mode. In any case, that method of averaging is to be preferred which is not only mathematically sound and self-consistent, but also is not too complicated to be readily understood and kept in mind. That is, it is essential that, as a working concept, the average be simple as well as accurate. Consequently, our discussion of averages will proceed from the familiar forms. Moreover, since we wish to obtain an average applicable to few commodities, ten or twenty, as well as to many commodities, fifty to a thousand, we shall discard the median and mode at the outset. The median and mode are unsatisfactory for the reason that they can be determined only within wide limits unless the items are very numerous. Further, even for a large number of items, the median and mode have serious defects for our present purpose:[1] these two averages are influenced only slightly, if at all, by substantial changes among the relatives, and the mode may be indeterminate. Our discussion of averages starts, therefore, with the assumption that the arithmetic, harmonic, and geometric means are more appropriate for our purpose than are other and less familiar averages.

The price relatives to be averaged are, as we have said, independent of the units of quantity with respect to which prices are expressed. This fact does not mean, however, that an average corresponding to a useful concept can be obtained without taking into account the aggregate quantities with which the relatives are associated. That is, in general we can not legitimately assume that it is possible to obtain a useful average of price relatives without weighting the relatives in proportion to quantities. Further, since the quantities of various commodities are expressed in terms of incommensurable units — bushels, tons, yards, feet, number — and since, from the point of view of economics, the only unit which all quantities of all commodities have in common is value, it is necessary to use weights based upon values.

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[1] For other purposes (such as the measurement of seasonal variation) the median or mode may, of course, be more appropriate than the arithmetic, harmonic and geometric means.

The formulas of the weighted arithmetic (A), harmonic (H), and geometric (G) means for period $n$ relative to period 1 (denoted by the subscripts), for $m$ commodities and weights, indicated by $W$, are:

$$A_{n\cdot 1}=\frac{\left(\frac{p'_n}{p'_1}\right)W'+\left(\frac{p''_n}{p''_1}\right)W''+\cdots+\left(\frac{p_n^{(m)}}{p_1^{(m)}}\right)W^{(m)}}{W'+W''+\cdots+W^{(m)}}$$

$$=\frac{\Sigma\left(\frac{p_n}{p_1}\right)W}{\Sigma W}$$

$$H_{n\cdot 1}=\frac{W'+W''+\cdots+W^{(m)}}{\left(\frac{p'_1}{p'_n}\right)W'+\left(\frac{p''_1}{p''_n}\right)W''+\cdots+\left(\frac{p_1^{(m)}}{p_n^{(m)}}\right)W^{(m)}}$$

$$=\frac{\Sigma W}{\Sigma\left(\frac{p_1}{p_n}\right)W}$$

$$G_{n\cdot 1}=\left[\left(\frac{p'_n}{p'_1}\right)^{W}\cdot\left(\frac{p''_n}{p''_1}\right)^{W''}\cdots\left(\frac{p_n^{(m)}}{p_1^{(m)}}\right)^{W^{(m)}}\right]^{\frac{1}{\Sigma W}}$$

$$=\Pi\left(\frac{p'_n}{p'_1}\right)^{\frac{W}{\Sigma W}}$$

It will be noticed that the arithmetic and harmonic means are both based, in part, on the mathematical process of summation, and hence are related to each other, while the geometric mean is based entirely on the mathematical process of multiplication.

In the numerical illustration with which we started, the prices of wheat and pig iron in two periods, suppose that the quantities correlative to the prices of wheat are 10,000 and 10,000 bushels ($q_1$ and $q_2$) and of pig iron are 1000 and 500 tons ($q_1$ and $q_2$), respectively, for the two periods. By combining these quantities with the corresponding prices ($pq$) we get four sets of weights based upon values, if we use the data of the two periods only, for the price relatives of each commodity in this binary comparison. These values and weights are:

| *Commodity* | *I* $p_1q_1$ | *II* $p_1q_2$ | *III* $p_2q_1$ | *IV* $p_2q_2$ |
|---|---|---|---|---|
| Wheat | $12,500 | $12,500 | $15,000 | $15,000 |
| Pig iron | 25,000 | 12,500 | 20,000 | 10,000 |
| Aggregate | 37,500 | 25,000 | 35,000 | 25,000 |
| | $p_1q_1/\Sigma p_1q_1$ | $p_1q_2/\Sigma p_1q_2$ | $p_2q_1/\Sigma p_2q_1$ | $p_2q_2/\Sigma p_2q_2$ |
| Wheat | $\frac{1}{3}$ or $.33\frac{1}{3}$ | $\frac{1}{2}$ or .50 | $\frac{3}{7}$ or $.42\frac{6}{7}$ | $\frac{3}{5}$ or .60 |
| Pig iron | $\frac{2}{3}$ or $.66\frac{2}{3}$ | $\frac{1}{2}$ or .50 | $\frac{4}{7}$ or $.57\frac{1}{7}$ | $\frac{2}{5}$ or .40 |

The averages of the price of wheat and the price of pig iron computed forward (for period 2 relative to period 1), weighted in four different ways, are:

| | *I* | *II* | *III* | *IV* |
|---|---|---|---|---|
| Arithmetic mean. | $\frac{14}{15}$ or .9333 | 1 or 1.0000 | $\frac{34}{35}$ or .9714 | $\frac{26}{25}$ or 1.0400 |
| Geometric mean.. | .9158 | .9798 | .9518 | 1.0203 |
| Harmonic mean.. | $\frac{9}{10}$ or .9000 | $\frac{24}{25}$ or .9600 | $\frac{14}{15}$ or .9333 | 1 or 1.0000 |

The averages of the price of wheat and the price of pig iron computed backward (for period 1 relative to period 2), weighted in the same four ways, are:

| | *I* | *II* | *III* | *IV* |
|---|---|---|---|---|
| Arithmetic mean . | $\frac{10}{9}$ or 1.1111 | $\frac{25}{24}$ or 1.0417 | $\frac{15}{14}$ or 1.0714 | $\frac{8}{8}$ or 1.0000 |
| Geometric mean.. | 1.0920 | 1.0206 | 1.0506 | .9801 |
| Harmonic mean.. | $\frac{15}{14}$ or 1.0714 | 1 or 1.0000 | $\frac{35}{34}$ or 1.0294 | $\frac{25}{26}$ or .9615 |

Certain features of the averages in the preceding tables are evident. First, when the base and type of average remain constant but different weights are used the results differ substantially from each other. Second, when the base and weights remain constant but different types of average are used, the arithmetic means are always larger than the geometric means and the latter are always larger than the harmonic means. Third, when the weights remain constant and the averages computed forward and backward are compared we find that:

(*a*) The geometric means give consistent results, that is, the products of the geometric means with constant weights, computed forward and backward, always equal one.

> (*b*) The arithmetic means, however, give inconsistent results, that is, the product of the two arithmetic means with constant weights, computed forward and backward, are not reciprocals of each other, but their product is always greater than one.
>
> (*c*) The harmonic means, likewise, give inconsistent results, that is, the product of the two harmonic means with constant weights, computed forward and backward, are not reciprocals, but their product is always less than one.
>
> (*d*) If we take the geometric mean of the arithmetic and harmonic means with constant weights, computed forward, we get consistent results with the geometric mean of the arithmetic and harmonic means, with the same constant weights, computed backward, that is, their product always equals one.

In attacking the problem of measuring the average change of the prices of two or more commodities between two periods we confront at the outset two difficulties, illustrated by the example just considered. These difficulties are:

> (*a*) The selection of weights in cases for which, in actual practice, weights and relatives may or may not be correlated.
>
> (*b*) The selection of a method of averaging which will be self-consistent, from the available methods which do not always give

reciprocal values when computed forward and backward.

In the problem of averaging relative prices — the problem of index numbers — we assume that we have as our equipment, first, actual prices; second, corresponding quantities; third, price relatives; fourth, value weights, and fifth, the familiar averages — the arithmetic, harmonic and geometric means.

We define an index number as an average of relatives, computed from actual economic data — a limitation which bars negatives, zeros, and infinites. We define an average as a function which has the properties that (*a*) if all the items averaged are equal to each other the average will be equal to each one of them also, and (*b*) if the items averaged are not all equal to each other the average will lie between the largest and smallest items. As we have said it is not our object to test every possible function having these two properties, but merely the averages corresponding to familiar mathematical processes and concepts — the arithmetic, geometric, and harmonic averages,[1] all of which possess the general properties just specified.

## I. The Problem of Measuring the Average Change of Prices between Two Periods

In searching for an answer to the question,

---

[1] The "aggregative" average is a special case of the arithmetic average.

"What is the average change of prices between two periods?" we encountered the following difficulties:

(*a*) Uncertainty as to the direction and amount of bias introduced by weights constructed from the pairs of products of the various prices and quantities for the two periods. One element in the weights, prices, always contributes to the positive or negative correlation of weights and price relatives; the other element, quantities, may or may not contribute to the correlation. In any case, the effect of the weights upon the resulting average of relative prices depends upon the correlation or lack of correlation between the weights, $pq/\Sigma pq$ and the relatives, $p_2/p_1$. The bias introduced on account of the correlation between weights and relatives to which we apply them will be called bias of the correlation of weights and associated relatives, or more shortly, *weight correlation bias.*[1]

---

[1] I depart from Fisher in that I assume that we may in actual practice have any sort of correlation between prices and quantities. Thus, in the first edition of his text, Appendix I, page 384, he says:

"In the text, systems I and II were summarily lumped together as practically the same, and likewise systems III and IV were lumped together. . . . Any index number under the system of weighting III *must* be larger than under weighting I. . . . In the 'short' step between III and IV, on the other hand, there is uncertainty. Any index number under III may be greater or less than under IV and may even *possibly* happen under very unusual circumstances to be *much* (*b*) Inconsistent results for the arithmetic (and harmonic) averages of price relatives with fixed weights when we shift the base of the relatives from one to the other of the pair of periods being compared (this inconsistency does not occur for the geometric mean, median, and mode). That is to say, bias will be introduced

larger or smaller. It is a fair lottery. The high price relatives may draw either heavy weights or light weights with an even chance each way, as, likewise, may the low price relatives. The net effect will *probably* be an almost complete offsetting so that the final index number (IV) will *probably* be close to III and may be either slightly above or below.

According to my notion this begs the question. In actual practice both high positive and high negative correlation frequently occurs between prices and quantities. Fisher says (footnote, page 384), "It is, of course, conceivable that there is a correlation between the prices and quantities but this may be in either direction according as the prime mover is supply or demand." All the work on statistical supply and demand curves of H. L. Moore and others indicates that in actual practice there is usually correlation between prices and quantities. Fisher continues, "Investigation of some New York Stock Exchange prices shows the same absence of correlation." My investigations bring me to precisely the opposite conclusion — that is to say, when prices are moving upward rapidly, the largest volume of sales occurs in those stocks for which prices are skyrocketing. At other times, the largest volume of sales occurs in those stocks for which prices are falling most rapidly.

In the case of the twelve crops for which I computed an index number of prices, Fisher necessarily admits the fact that there is an inverse correlation between quantity and prices. He adds that in this case "it is noteworthy that the effect on the curves given in Chapter XI is almost negligible." It is negligible when pairs of years only are compared. For a series of years it actually results in a wider and wider divergence of the fixed base and chain indexes.

TABLE 1. — NUMBER OF EQUAL CONTRIBUTIONS TO WEIGHT CORRELATION BIAS IN VARIOUS WEIGHTED AVERAGES (USED IN EXPRESSING THE AVERAGE CHANGE OF PRICES BETWEEN TWO PERIODS) ON THE BASIS OF EACH OF THE FOLLOWING THREE ASSUMPTIONS CONCERNING CORRELATION BETWEEN $p$ AND $q$: (1) NO CORRELATION, (2) PERFECT POSITIVE CORRELATION, AND (3) PERFECT NEGATIVE CORRELATION

| VALUE WEIGHTS | (1) No Correlation between $p$'s and $q$'s[1] | | | | | | (2) Perfect Positive Correlation[2] | | | | | | (3) Perfect Negative Correlation[3] | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (g) Geometric | | (h) Harmonic | | (a) Arithmetic | | (g) Geometric | | (h) Harmonic | | (a) Arithmetic | | (g) Geometric | | (h) Harmonic | | (a) Arithmetic | |
| | U | D | U | D | U | D | U | D | U | D | U | D | U | D | U | D | U | D |
| $p$'s for base year: $q$'s for base year (I) | | I | | I | | I | | II | | II | | II | I | I | I | I | I | I |
| $q$'s for given year (II) | | I | | I | | I | I | I | I | I | I | I | | II | | II | | II |
| $p$'s for given year: $q$'s for base year (III) | I | | I | | I | | I | I | I | I | I | I | II | | II | | II | |
| $q$'s for given year (IV) | I | | I | | I | | II | | II | | II | | I | I | I | I | I | I |

[1] In this case the bias "tallies" are on account of the price element in weights.

[2] In this case to the tallies of section (1) *are added* the "up" tallies for $q$ for the given year and the "down" tallies for $q$ for the base year.

[3] In this case to the tallies of section (1) *are added* the "down" tallies for $q$ for the given year and the "up" tallies for $q$ for the base year.

into an arithmetic or harmonic average whether one or the other base be selected. This bias will be called bias of the arbitrary base period, or more shortly, *base bias*.

The difficulties on account of the bias of (*a*) weight correlation and (*b*) base period were met as follows:

In order to ascertain the effect of weights with certainty three *special cases* of prices and associated quantities were considered. These are:

(1) The price relatives, $p_2/p_1$, are assumed to be correlated with the associated weights, $pq/\Sigma pq$, only through the price element appearing in each.

(2) Prices and quantities are assumed to contribute equally in magnitude and direction to the correlation between the price relatives, $p_2/p_1$, and their weights, $pq/\Sigma pq$.

(3) Prices and quantities are assumed to contribute equally in magnitude, but in opposite directions to the correlation between the price relatives, $p_2/p_1$, and their weights, $pq/\Sigma pq$.

For each of the three means (geometric, harmonic, and arithmetic) in each of the three special cases, a summary (Table 1) was made showing the number of equal contributions (up or down) to weight correlation bias by prices and quantities for each of the four combinations ($p_1q_1$, $p_1q_2$, $p_2q_1$, and $p_2q_2$) appearing in the weights.

These four combinations are designated I, II, III and IV.

It is obvious that it is possible to select *pairs* of geometric means from case (1) of Table 1 in which one member of each pair will have an upward weight correlation bias and the other member will have a downward weight correlation bias. These pairs, four in all, appear in the second column of Table 2. It is possible to select two such pairs of geometric means from case (2) and two from case (3) with opposite weight correlation bias. These pairs appear in the third and fourth columns of Table 2. Inspection of the pairs of geometric means appearing for the three cases reveals the fact that two pairs (I*g* and IV*g*) and (II*g* and III*g*) are common to all. Consequently, we conclude that a geometric mean of relatives either with a combination of weights I and IV or of weights II and III would have no weight correlation bias whether the correlation between prices and quantities be perfect positive, zero, or perfect negative. Since these weights result in an average with no weight correlation bias for the three pivotal cases of correlation between $p$ and $q$ it is likely that the same result would be achieved for the intermediate cases of correlation or a mixture of the three pivotal cases. Consequently, we conclude that these geometric averages, which we have numbered i and ii, are superior, for measuring the average change of

Table 2. — The Combinations (Using the Geometric Mean) of Two Geometric Means ($g$) that will Result in Formulas without Weight Correlation Bias for Each of the Three Special Cases of Table 1

| Combinations* appearing in all three Cases | Combinations* appearing in the Case Indicated | | |
|---|---|---|---|
| | Case (1) | Case (2) | Case (3) |
| (i) (I$g$ & IV$g$) | (I$g$ & III$g$) | (I$g$ & IV$g$) | (I$g$ & IV$g$) |
| (ii) (II$g$ & III$g$) | (I$g$ & IV$g$) | (II$g$ & III$g$) | (II$g$ & III$g$) |
| | (II$g$ & III$g$) | | |
| | (II$g$ & IV$g$) | | |

---

* In the notation the numerals I, II, III and IV refer, respectively, to the weights in terms of $p_1q_1$, $p_1q_2$, $p_2q_1$ and $p_2q_2$ which are applied to the relatives appearing in the geometric mean.

### TABLE 3.— THE COMBINATIONS (USING THE GEOMETRIC MEAN) OF ONE HARMONIC MEAN (*h*) AND ONE ARITHMETIC MEAN (*a*) THAT WILL RESULT IN FORMULAS WITHOUT WEIGHT CORRELATION BIAS FOR EACH OF THE THREE SPECIAL CASES OF TABLE 1

| Combinations* appearing in all three cases | Combinations* not appearing in all three cases but which appear in the case indicated | | |
|---|---|---|---|
| | Case (1) | Case (2) | Case (3) |
| (iii) (I*h* & IV*a*) | (I*h* & III*a*) | (II*h* & II*a*) | (I*h* & I*a*) |
| (iv) (II*h* & III*a*) | (II*h* & IV*a*) | (III*h* & III*a*) | (IV*h* & IV*a*) |
| (v) (III*h* & II*a*)† | (III*h* & I*a*) | | |
| (v) (IV*h* & I*a*)† | (IV*h* & II*a*) | | |

---

[*] In the notation the numerals I, II, III and IV refer, respectively, to the weights in terms of $p_1q_1$, $p_1q_2$, $p_2q_1$, $p_2q_2$ which are used either in the harmonic (*h*) or arithmetic (*a*) average.

† These formulas are identical.

prices between two periods, to geometric averages with weights which do not satisfy the condition.

Further, since any geometric mean with fixed weights computed forward is the reciprocal of the geometric mean with fixed weights computed backward the base reversal test is satisfied by the geometric averages i and ii. Any other average is considered to be logically inferior to these — for the purpose of measuring the average change of prices between two periods — if (*a*) it is not free from weight correlation bias and (*b*) it does not satisfy the base reversal test, *i.e.*, the reversal of the base should result in reciprocal index numbers. That is, these geometric means, being free both from weight correlation bias and base bias, are logically superior to all averages not having these features.

Other averages than the geometric mean can be found to satisfy both of the essential requirements just specified. The arithmetic and harmonic means, however, as we have seen, do not satisfy the base reversal test. In fact, it is easily proven that the product of the arithmetic means computed forward and backward is *always* greater than one and the product of the two harmonic means computed forward and backward is *always* less than one.[1] That is to say (using the notation defined on page 5):

---

[1] One assumption is implicit: the relatives averaged are not all equal.

$$A_{2\cdot1}\cdot A_{1\cdot2} > 1$$
$$H_{2\cdot1}\cdot H_{1\cdot2} < 1$$

But, it is also easily proven, assuming fixed weights, that[1]

$$A_{2\cdot1}\cdot H_{1\cdot2} = 1$$
$$(A_{2\cdot1}\cdot H_{2\cdot1})\ (A_{1\cdot2}\cdot H_{1\cdot2}) = 1$$

The equality just stated means that the product of the arithmetic mean and harmonic mean with the *same weights in all means* satisfies the base reversal test. Therefore, to obtain averages (*a*) free from weight correlation bias and (*b*) which *may* satisfy the base reversal test we may take the square root of the product of an arithmetic mean and harmonic mean from Table 1, choosing pairs of averages in which the weight correlation bias of one member of the pair will be opposite to that of the other member of the pair. The combinations of one arithmetic mean, and one harmonic mean, chosen in this fashion, are given for each of the three special cases in Table 3. There are three combinations, iii, iv and v, which are common to all three pivotal cases, and hence are free from weight correlation bias in these cases. Further, it is easily demonstrated that the combinations iii, iv and

---

[1] The proof follows:

$$A_{2\cdot1} = \frac{\Sigma\left(\frac{p_2}{p_1}\right)W}{\Sigma W} \text{ and } H_{1\cdot2} = \frac{\Sigma W}{\Sigma\left(\frac{p_2}{p_1}\right)W}$$

$$\therefore\ A_{2\cdot1}\cdot H_{1\cdot2} = 1$$

v satisfy the base reversal test. Therefore, the five combination averages of Tables 2 and 3 are (*a*) free from weight correlation bias and (*b*) satisfy the base reversal test.

It will be noticed from Table 4 that formula v, the "ideal," is identical with the geometric mean of (I*a*, II*a*, III*h* and IV*h*). That is to say, the "ideal" index of prices may be considered to be the geometric mean of four constituent indexes of prices, two of these constituents being arithmetic means of relatives, with weights $p_1q_1/\Sigma p_1q_1$ and $p_1q_2/\Sigma p_1q_2$, respectively, and the other two constituents being harmonic mean of relatives with weights $p_2q_1/\Sigma p_2q_1$ and $p_2q_2/\Sigma p_2q_2$, respectively. In still other words, the "ideal" index of prices, in a binary comparison, is the result of using as weights for the relatives averaged all possible combinations of prices ($p_1$ and $p_2$) with the associated quantities ($q_1$ and $q_2$) of the two periods.

Combination formulas i to v are, however, not the only ones that can be obtained from Table 1 to satisfy our two tests. We may combine four geometric means, or two harmonic means with two arithmetic means, in such a way as to have offsetting weight bias for each of the three pivotal cases. There is, of course, only one combination of four geometric means, *i.e.* (I*g*, II*g*, III*g* and IV*g*), which satisfies the condition. This formula will be numbered vi.

Table 4 gives the results of combining two

TABLE 4. — THE COMBINATIONS (USING THE GEOMETRIC MEAN) OF TWO HARMONIC MEANS ($h$) AND TWO ARITHMETIC MEANS ($a$) THAT WILL RESULT IN FORMULAS WITHOUT WEIGHT CORRELATION BIAS FOR EACH OF THE THREE SPECIAL CASES OF TABLE 1.

| Combinations* appearing in all three cases | Combinations* not appearing in all three cases but which appear in the case indicated | | |
|---|---|---|---|
| | Case (1) | Case (2) | Case (3) |
| (v) (III & IV$h$, I & II$a$)† | (II & IV$h$, I & IV$a$) | | |
| | (II & IV$h$, II & III$a$) | | |
| | (II & IV$h$, II & IV$a$) | | |
| (vii) (I & II$h$, III & IV$a$)† | (I & III$h$, I & III$a$) | (I & II$h$, II & IV$a$) | (I & II$h$, I & III$a$) |
| (viii) (I & III$h$, II & IV$a$)† | (I & III$h$, I & IV$a$) | (I & III$h$, III & IV$a$) | (I & III$h$, I & II$a$) |
| (viii) (I & IV$h$, I & IV$a$)† | (I & III$h$, II & III$a$) | (II & IV$h$, I & III$a$) | (II & IV$h$, III & IV$a$) |
| (ix) (II & III$h$, II & III$a$)† | (II & III$h$, I & III$a$) | | |
| (ix) (II & IV$h$, I & III$a$)† | (II & III$h$, II & IV$a$) | | |
| (x) (I & IV$h$, II & III$a$) | (I & IV$h$, I & III$a$) | (III & IV$h$, I & III$a$) | (III & IV$h$, II & IV$a$) |
| (xi) (II & III$h$, I & IV$a$) | (I & IV$h$, II & IV$a$) | | |

* In the notation the numerals I, II, III and IV refer, respectively, to the weights in terms of $p_1q_1$, $p_1q_2$, $p_2q_1$ and $p_2q_2$ which are applied to the relatives appearing in the geometric mean.

† These cases are identical with the six possible combinations, two at a time, of the four pairs of formulas in column 1 of Table 3. The third and fourth formulas in the list of column 1 of Table 4 are identical, the fifth and sixth are identical and the first is identical with the ideal formula. In Table 4, therefore, there are really only two new formulas (in addition to those of Table 3) and these (x and xi) do not satisfy the base reversal test.

arithmetic means and two harmonic means in such a way as to have offsetting weight correlation bias for the three pivotal cases. In the first column of that table there appear eight formulas common to the three cases. Of these eight formulas six are combinations taken two at a time of the four formulas (iii, iv, v and v) in column 1 of Table 3. These six combinations, of course, satisfy the base reversal test since they are made up of pairs of means which satisfy that test.

Formulas x and xi, however, do not satisfy the base reversal test and should be eliminated from our list of averages which are (*a*) free from weight correlation bias, (*b*) satisfy the base reversal test and, therefore, are logically superior to other averages for measuring the average change of prices between two periods. Consequently, only nine formulas, i to ix, have been found which satisfy the two conditions (*a*) and (*b*) and of these, four, vi to ix, are combinations of pairs of formulas i to v.

To recapitulate, the formulas we have found that satisfy tests (*a*) and (*b*) are:

$$\text{(i)} \qquad \Pi \left( \frac{p_2}{p_1} \right)^{\frac{1}{2}(p_2q_2/\Sigma p_2q_2 + p_1q_1/\Sigma p_1q_1)}$$

$$\text{(ii)} \qquad \Pi \left( \frac{p_2}{p_1} \right)^{\frac{1}{2}(p_2q_1/\Sigma p_2q_1 + p_1q_2/\Sigma p_1q_2)}$$

$$\text{(iii)} \qquad \left\{ \frac{\Sigma p_1 q_1}{\Sigma\left(\dfrac{p_1}{p_2}\right) p_1 q_1} \cdot \frac{\Sigma\left(\dfrac{p_2}{p_1}\right) p_2 q_2}{\Sigma p_2 q_2} \right\}^{\frac{1}{2}}$$

$$\text{(iv)} \qquad \left\{ \frac{\Sigma p_1 p_2}{\Sigma\left(\dfrac{p_1}{p_2}\right) p_1 q_2} \cdot \frac{\Sigma\left(\dfrac{p_2}{p_1}\right) p_2 q_1}{\Sigma p_2 q_1} \right\}^{\frac{1}{2}}$$

$$\text{(v)} \qquad \left( \frac{\Sigma p_2 q_2}{\Sigma p_1 q_2} \cdot \frac{\Sigma p_2 q_1}{\Sigma p_1 q_1} \right)^{\frac{1}{2}}$$

(vi) Geometric mean of i and ii
(vii) Geometric mean of iii and iv
(viii) Geometric mean of iii and v
(ix) Geometric mean of iv and v

It will be interesting to examine the actual results obtained by computing index numbers of prices, according to various formulas for averages, for a special illustrative case. We have selected for this purpose the case used by Irving Fisher in the *Making of Index Numbers*[1] in which annual prices and quantities for 36 commodities for 1913–18 are utilized. Table 5 presents the price index numbers computed by using the weighted geometric, harmonic and arithmetic averages. Certain of these averages (I*a* or III*h*, II*a* or IV*h*) may be expected, in theory, to give fairly accurate results because the base bias of the formulas is opposite to the weight correlation bias. In all the other index numbers of Table 5 base bias does not tend to be neutralized or offset by weight correlation bias.[1]

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[1] Pp. 498–504.

Table 6 presents a comparison of the nine index numbers satisfying the (*a*) weight correlation and (*b*) base reversal tests, for the same illustrative case of 36 commodities. These index numbers give remarkably consistent results. The range of variation of the fixed base index numbers for the nine formulas 1914–18 is as follows:

| | *1914* | *1915* | *1916* | *1917* | *1918* |
|---|---|---|---|---|---|
| Range | 100.09 | 99.87 | 113.83 | 161.37 | 176.63 |
| | to | to | to | to | to |
| | 100.12 | 100.06 | 114.59 | 163.10 | 177.80 |

The greatest range is in 1917 when it amounts to slightly over one per cent. The two index numbers, x and xi, which do not satisfy the base reversal test have values within the range given for the nine index numbers which do satisfy that test.

Two generalizations may, apparently, be drawn from the results obtained from the special illustrative case of 36 commodities. First, index numbers which satisfy both the weight correla-

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[1] In the illustrative case of 36 commodities here considered, the correlation between relatives and weights is principally through prices.

TABLE 5. — COMPARISON OF WEIGHTED GEOMETRIC, HARMONIC, AND ARITHMETIC AVERAGES FOR A SPECIAL ILLUSTRATIVE CASE†

| Formula | Fisher's Formula Number | Base | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|---|---|
| I*a* or III*h* | 3 or 17 or 53 | Fixed | 99.93 | 99.67 | 114.08 | 162.07 | 177.87 |
| | | Chain | 99.93 | 100.13 | 113.82 | 162.44 | 178.56 |
| II*a* or IV*h* | 5 or 17 or 54 | Fixed | 100.32 | 100.10 | 114.35 | 161.05 | 177.43 |
| | | Chain | 100.32 | 100.33 | 114.83 | 162.02 | 178.43 |
| III*a* | 7 | Fixed | 100.55 | 101.77 | 117.77 | 180.53 | 186.98 |
| IV*a* | 9 | Fixed | 100.93 | 102.33 | 118.29 | 180.72 | 187.18 |
| | | Chain | 100.93 | 102.10 | 122.41 | 180.40 | 205.56 |
| I*h* | 13 | Fixed | 99.26 | 97.84 | 111.01 | 147.19 | 168.59 |
| | | Chain | 99.26 | 98.45 | 108.19 | 148.14 | 157.78 |
| II*h* | 15 | Fixed | 99.65 | 98.11 | 111.02 | 144.97 | 166.85 |
| I*g* | 23 | Fixed | 99.61 | 98.72 | 111.45 | 154.08 | 173.30 |
| | | Chain | 99.61 | 99.28 | 110.91 | 155.03 | 166.93 |
| II*g* | 25 | Fixed | 99.99 | 99.07 | 112.58 | 152.45 | 172.37 |
| III*g* | 27 | Fixed | 100.25 | 100.67 | 115.82 | 170.82 | 182.45 |
| IV*g* | 29 | Fixed | 100.63 | 101.17 | 116.26 | 170.44 | 182.41 |

† From Fisher's *The Making of Index Numbers*, pp. 498–504.

tion and base reversal tests give highly consistent results. Second, satisfaction of the weight correlation test is practically much more essential for such consistency, than satisfaction of the base reversal test — since formulas x and xi give results highly similar to those of formulas satisfying the base reversal test.

The formulas from Tables 2, 3, 4 and the additional formula coming from combination of four geometric means with different sets of weights gave us nine formulas for price indexes in a binary comparison, each of the price indexes being without (*a*) weight correlation bias and (*b*) base bias.

One and only one of the nine formulas for price indexes (v) satisfies a third test, that is to say, the factor reversal test, which may properly be set up for the selection, from the nine, of the logically best index. The factor reversal test states that the

(Index of Prices) (Correlative Index of Quantities) = Ratio of Values

or $$P_{21} \cdot Q_{21} = V_{21}$$

where the index of quantities ($Q_{21}$) is obtained from the index of prices ($P_{21}$) by interchanging the $p$'s and $q$'s and the index of values is the ratio $\Sigma p_2 q_2 / \Sigma p_1 q_1$. Other of the nine indexes than v may be shown, in actual practice, to very nearly fulfill the factor reversal test, but such

Table 6. — Comparison of the Nine Index Numbers, for a Special Illustrative Case, Satisfying the (*a*) Weight Correlation and (*b*) Base Reversal Tests, and Two Index Numbers (X and XI) which Satisfy only the Weight Correlation Test

| Formula | | Fisher's Formula Number | Base | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|---|---|---|
| (i) | (I*g* and IV*g*)† | 23 and 29 or 123 | Fixed | 100.12 | 99.94 | 113.83 | 162.05 | 177.80 |
| | | | Chain | 100.12 | 100.24 | 114.63 | 162.75 | 178.87 |
| (ii) | (II*g* and III*g*)† | 25 and 27 or 125 | Fixed | 100.12 | 99.87 | 114.19 | 161.37 | 177.34 |
| | | | Chain | 100.12 | 100.24 | 114.33 | 162.18 | 178.36 |
| (iii) | (I*h* and IV*a*)† | 13 and 9 or 109 | Fixed | 100.09 | 100.06 | 114.59 | 163.10 | 177.64 |
| (iv) | (II*h* and III*a*)† | 15 and 7 or 107 | Fixed | 100.10 | 99.92 | 114.35 | 161.78 | 176.63 |
| (v) | (III & IV*h*, I & II*a*)† | 353 | Fixed | 100.12 | 99.89 | 114.21 | 161.56 | 177.65 |
| | | Ideal | Chain | 100.12 | 100.23 | 114.32 | 162.23 | 178.49 |

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† See Fisher's *The Making of Index Numbers*, pp. 498–518.

* These index numbers were computed as geometric means of constituent factors as given in Fisher's *The Making of Index Numbers*, pp. 498–518.

TABLE 6.— *Continued*

| Formula | Fisher's Formula Number | Base | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|---|---|
| (vi) (I*g* & II*g* & III*g* & IV*g*)* | 123 and 125 | Fixed<br>Chain | 100.12<br>100.12 | 99.90<br>100.24 | 114.01<br>114.48 | 161.71<br>162.47 | 177.57<br>178.62 |
| (vii) (I & II*h*, III & IV*a*)* | 109 and 107 | Fixed | 100.10 | 99.99 | 114.47 | 162.44 | 177.13 |
| (viii) (I & III*h*, II & IV*a*)* | 109 and 105 or 109 and 353 | Fixed | 100.11 | 99.97 | 114.40 | 162.33 | 177.65 |
| (ix) (II & III*h*, II & III*a*)* | 105 and 107 or 353 and 107 | Fixed | 100.11 | 99.90 | 114.28 | 161.67 | 177.14 |
| (x) (I & IV*h*, II & III*a*)* | (13, 19, 5, 7) or (13, 54, 54, 7) | Fixed | 100.11 | 99.94 | 114.34 | 162.03 | 177.49 |
| (xi) (II & III*h*, I & IV*a*)* | (15, 17, 3, 9) or (15, 53, 53, 9) | Fixed | 100.11 | 99.93 | 114.34 | 161.96 | 177.30 |

*These index numbers were computed as geometric means of constituent factors as given in Fisher's *The Making of Index Numbers*, pp. 498-518.

formulas, as we have said, do not, in general, fulfill it perfectly.

We conclude that formula v for prices is logically the best answer that we can give to the question, "What is the average change of prices between two periods or places?" Moreover, the correlative or companion formula for quantities (obtained by substituting $q$ for $p$ and $p$ for $q$ in formula v) is logically the best answer that we can give to the question, "What is the average change of quantities between two periods or places?"

We may point out in this connection that if we take *any* average of prices ($P_{2\,1}$) it is possible to find a *formula* in terms of the $p$'s and $q$'s which will satisfy both the base reversal test and the factor reversal test. This formula[1] is

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[1] See Fisher's the *Making of Index Numbers*, p. 399.

*Note on Fisher's Method of Building Up Formulas to Satisfy His Factor Reversal Text* (see *The Making of Index Numbers*, p. 396):

Let $P_{2\,1}$ be any price index for period 2 relative to period 1, and $V_{2\,1}$ be the ratio of the aggregate values of period 2 to period 1. Then the "factor antithesis" of $P_{2\,1}$ is $V_{2\,1}/Q_{2\,1}$, where $Q_{2\,1}$ is obtained by interchanging the $p$'s and $q$'s in $P_{2\,1}$.

If we now "rectify" $P_{2\,1}$ by taking the geometric means of $P_{2\,1}$ and its "factor antithesis" we get

$$\left(\frac{P_{2\,1} \cdot V_{2\,1}}{Q_{2\,1}}\right)^{\frac{1}{2}}$$

which, by its method of construction, will satisfy the factor reversal test. We now have a *new formula* which we propose to use as *an index of prices*. This new formula is the geometric mean of

$$P_{2\,1} \text{ and } V_{2\,1}/Q_{2\,1}$$

$$\left(\frac{P_{1\,2}\cdot Q_{2\,1}\cdot(\Sigma p_2q_2)^2}{P_{2\,1}\cdot Q_{1\,2}\cdot(\Sigma p_1q_1)^2}\right)^{1/4}$$

Although this formula satisfies the two tests it is, in general, neither an average of prices nor an average of quantities. The formula is not, therefore, of use in obtaining an answer to the question, "What is the average change of prices or quantities between two periods or places?"

In connection with the "ideal" index (v) we may point out that it is simultaneously:

the square root of the product ($\mathrm{I}a\cdot\mathrm{II}a$);
the square root of the product ($\mathrm{III}h\cdot\mathrm{II}a$);
the square root of the product ($\mathrm{I}a\cdot\mathrm{IV}h$);

The first factor, $P_{2\,1}$, is the original index of prices with which we started. The second factor $V_{2\,1}/Q_{2\,1}$ is also supposed to be an *index of prices*. It is obtained by dividing the index of values $V_{2\,1}$, by the index of quantities $Q_{2\,1}$, correlative to our original index of prices.

May we properly call the factor $V_{2\,1}/Q_{2\,1}$ an index of prices? We agree with Fisher (p. 3) that we cannot, for, as he says, "an *index number* of the prices of a *number* of commodities is an *average* of their price relatives." In general, the factor cannot be expressed as a function of price relatives and cannot be called an average of prices.

If the first factor is a proper index of prices and the second factor is not, the mean of the two factors $(P_{2\,1}.\ V_{2\,1}/Q_{2\,1})^{\frac{1}{2}}$ cannot, in general, be called an index of prices. Fisher, nevertheless, makes the finding and testing such *formulas* the central feature of his book. Further, when he compares the results obtained by using such formulas for the special case of 36 commodities 1913–18 he erroneously calls the formulas "index numbers" (Table 28, pp. 246–7), whereas many of the formulas are not proper averages and, therefore, not index numbers.

the square root of the product (III*h*·IV*h*);
the fourth root of the product (I*a*·II*a*·III*h*·IV*h*).

We have stated that only one index (v) among the nine exactly, and in general, satisfies the factor reversal test, but that the other indexes very nearly satisfy it. In connection with the latter part of the statement, *i.e.*, "that the other indexes nearly satisfy the factor reversal test," let us consider the price index constructed from geometric means of relatives (say, formula vi).

Consider the index, $\Pi\left(\frac{p_2}{p_1}\right)^W$, where

$W = \frac{1}{4}(p_1q_1/\Sigma p_1q_1 + p_1q_2/\Sigma p_1q_2 + p_2q_1/\Sigma p_2q_1 + p_2q_2/\Sigma p_2q_2)$ and the correlative index of quantities, $\Pi\left(\frac{q_2}{q_1}\right)^W$. The product of these two indexes is $\Pi\left(\frac{p_2q_2}{p_1q_1}\right)^W$. The numerator is the weighted product of the values $p_2q_2$ and the denominator is the weighted product of the values $p_1q_1$.

The question which we are considering is the *divergence* between the following ratios:

$$\frac{\text{the weighted product of } p_2q_2}{\text{the weighted product of } p_1q_1}$$

and $$\frac{\text{the aggregate of } p_2q_2}{\text{the aggregate of } p_1q_1}$$

We cannot tell in general what the divergence between the two ratios will be, but we can ascertain the divergence for a special case. Let us consider the special case of 36 commodities for 1913–18, used by Fisher, and compare the ratios of aggregate values with the ratios of weighted products (see Table 7).

TABLE 7. — VALUE RATIOS COMPARED WITH PRODUCT RATIOS FOR 36 COMMODITIES, 1913–18

| Year | Value Ratio | Product Ratios | | |
|---|---|---|---|---|
| | | Formula (vi) | Formula (i) | Formula (ii) |
| 1913 | 100.00 | 100.00 | 100.00 | 100.00 |
| 1914 | 99.45 | 99.43 | 99.42 | 99.45 |
| 1915 | 108.98 | 109.03 | 109.07 | 109.00 |
| 1916 | 135.75 | 135.56 | 135.37 | 135.74 |
| 1917 | 192.23 | 192.53 | 192.60 | 192.46 |
| 1918 | 222.71 | 222.47 | 222.26 | 222.68 |

It will be seen from the second and third columns of the table that in this special case the products of the indexes of prices with correlative indexes of quantities, using index formula vi, for the years 1914–18 is never more than 16/100 of one per cent from the value ratio obtained directly. The maximum deviation of the products of the price and quantity indexes according to formulas i and ii, respectively, are 28/100 and 12/100 of 1 per cent. Formulas i, ii and vi, in this special case, therefore, all come within 28/100 of one per cent of satisfying the factor reversal test for prices and quantities.

The percentage deviations just given are less than the deviations of the price indexes, computed from these formulas and the price index computed from formula v, which perfectly satisfies the factor reversal test. The maximum deviation of price index vi from price index v for the same year is 2 parts in 1142 or less than two tenths of one per cent. The deviations of indexes of prices as computed by formulas i, ii and vi from the indexes computed by formula v (the ideal) for the special case are as follows:

| Price Index | Actual Deviations from Price Index v | | | | |
|---|---|---|---|---|---|
| | 1914 | 1915 | 1916 | 1917 | 1918 |
| (i) | 0 | +.05 | −.38 | +.49 | +.15 |
| (ii) | 0 | −.02 | −.02 | −.19 | −.31 |
| (vi) | 0 | +.01 | −.20 | +.15 | −.08 |
| | Price Index v | | | | |
| (v) | 100.12 | 99.89 | 114.21 | 161.56 | 177.65 |

*We may justly characterize the price index* v *as logically superior to the other indexes that we have examined (and in fact superior to any other known index) for a binary comparison.*[1] The logical superiority of v to the other indexes of our group of nine results from the fact that v alone satisfies the factor reversal test. *But the other indexes very nearly satisfy that test.* The factor reversal test, therefore, merely gives us a basis for selecting one of a number of indexes, all of which give highly similar results (as shown by an illustrative case) in a binary comparison. That is to say, the factor reversal test enables us to select the logically best index from a number of logically good indexes which, when tested in actual cases, give highly similar results. In still other words, the factor reversal test selects one index from a number of indexes in the same homogeneous group, both logically and actually; it does not select a good index from a heterogeneous group of indexes. Finally, all of the tests refer to a binary comparison only.

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[1] The "ideal" index (v) is logically superior to other indexes for a binary comparison only. For a comparison of three or more periods each with the other, we shall show that it is impossible for an average to satisfy both the factor reversal

*Effect of weights on various items of any series.* Suppose that we consider a case of prices and associated quantities of the most general sort, *i.e.*, in which some of the price and quantity relatives are positively correlated, some are

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test and the circular test (the latter being the generalization for three or more periods of the base reversal test for two periods). In such a case — of partially conflicting criteria — where it is impossible for us to decide whether one criterion is more important than another, the formula which comes *closest* to satisfying all of the criteria, although it satisfies none exactly, appears to hold as strong or a stronger logical position than another formula which satisfies some of the criteria exactly and the others with a considerable margin of error.

negatively correlated, and some are correlated neither positively nor negatively. Assume that the formula (v) found above for prices and the companion formula for quantities are utilized in computing index numbers for this general case. We will call these index numbers $P$ and $Q$. The items of the series (the general case) may be divided into five subgroups, as follows:

For subgroup (1) $p_2/p_1 > P$ and $q_2/q_1 \geqq Q$
For subgroup (2) $p_2/p_1 > P$ and $q_2/q_1 < Q$
For subgroup (3) $p_2/p_1 < P$ and $q_2/q_1 \geqq Q$
For subgroup (4) $p_2/p_1 < P$ and $q_2/q_1 < Q$
For subgroup (5) $p_2/p_1 = P$ and $q_2/q_1 \overset{>}{\underset{<}{=}} Q$

For subgroup (1) of any series $p_2q_2/\Sigma p_2q_2 > p_1q_1/\Sigma p_1q_1$ and for subgroup (4) $p_2q_2/\Sigma p_2q_2 < p_1q_1/\Sigma p_1q_1$. Also, *there may be* a section (*a*) of the subgroup (2) for which the first of the above relationships between the weights holds true and a section (*b*) of the subgroup (3) for which the second relationship holds true. We conclude that:

For all items in subgroups (1), (2*a*), (3*b*) and (4) the use of given year weights will tend to result in a larger price index than that obtained from base year weights (and also larger than $P$, which presumably has no weight correlation bias).

For all items in subgroup (5) variation in weights will not tend to change the price index $P$.

For all items in subgroups (2*b*) and (3*a*) the use of given year weights will tend to result in a smaller price index than that obtained from base year weights (and also smaller than $P$, which presumably has no weight correlation bias).

For the whole series, the use of given year weights will actually result in a larger index than that obtained from the use of base year weights *unless*, first, there are one or more relatives in subgroup (2*b*) with negatively correlated weights, or/and in subgroup (3*a*) with positively correlated weights and, second, the downward effect of the weight bias for these relatives is more than sufficient in magnitude to offset the upward effect of the weight bias of the remainder of the series.

For subgroup (2) of any series we may easily prove that $p_2q_1/\Sigma p_2q_1 > p_1q_2/\Sigma p_1q_2$ and for subgroup (3) that $p_2q_1/\Sigma p_2q_1 < p_1q_2/\Sigma p_1q_2$. Also, *there may be* a section (*a*) of the subgroup (1) for which the first part of the above relationship between the weights holds true and a section (*b*) of the subgroup (4) for which the second part of the relationship holds true. We conclude that:

For all the items in subgroups (1*a*), (2), (3) and (4*b*) the use of given year prices and base year quantities in weights will tend to result in a larger price index than that obtained from the use of base year prices and given year quantities in weights (and also larger than $P$, which presumably has no weight correlation bias).

For all items in subgroup (5) variation in weights will not tend to change the price index $P$.

For all items in subgroups (1*b*) and (4*a*) the use of given year prices and base year quantities in weights will tend to result in a smaller price index than that obtained from the use of base year prices and given year quantities in weights (and also smaller than $P$ which presumably has no weight bias).

For the whole series, the use of given year prices and base year quantities will actually result in a larger index than that obtained from the use of base year prices and given year quantities in weights *unless*, first, there are one or more relatives in subgroup (1*b*) with negatively correlated weights or/and in subgroup (4*a*) with positively correlated weights and, second, the downward effect of the weight bias in these relatives is more than sufficient in magnitude to offset the upward effect of the weight bias of the remainder of the series.

## II. THE PROBLEM OF MEASURING THE AVERAGE CHANGE OF PRICES BETWEEN THREE OR MORE PERIODS

If $P_{2\,1}$, $P_{3\,1}$ and $P_{3\,2}$ are the best index numbers for comparing price changes between *pairs* of three periods (1, 2, 3) does it follow that they are the best to use for a *simultaneous comparison* *of each year with both other years*, such a comparison as we make when we plot three or more index numbers on the same chart?

First, it appears that if $P_{2\,1}$ represents the average of prices in period 2 relative to period 1 and $P_{3\,1}$ represents the average of prices in period 3 relative to period 1 then $P_{3\,1}/P_{2\,1}$ should equal $P_{3\,2}$. But this relationship will not hold in general for any index in which the weights of the relatives in the three values $P_{3\,1}$, $P_{2\,1}$ and $P_{3\,2}$ differ from each other. In other words, in order that the relationship $P_{3\,1}/P_{2\,1}=P_{3\,2}$ hold true in general we must have fixed weights. Thus, for the ideal index (v) to satisfy the circular text we should have

$$\left(\frac{\Sigma p_3q_1}{\Sigma p_1q_1}\cdot\frac{\Sigma p_3q_3}{\Sigma p_1q_3}\right)^{1/2}=\left(\frac{\Sigma p_2q_1}{\Sigma p_1q_1}\cdot\frac{\Sigma p_2q_2}{\Sigma p_1q_2}\cdot\frac{\Sigma p_3q_2}{\Sigma p_2q_2}\cdot\frac{\Sigma p_3q_3}{\Sigma p_2q_3}\right)^{1/2}$$

and
$$\frac{\Sigma p_3q_1}{\Sigma p_3q_2}=\frac{\Sigma p_2q_1}{\Sigma p_2q_3}\cdot\frac{\Sigma p_1q_3}{\Sigma p_1q_2}$$

It is evident, therefore, that formula v does not reduce to an identity and will not satisfy the circular test *unless* there is a certain special relationship between the *p*'s and *q*'s, *i.e.*, the relationship indicated by the preceding equation. If we assume (1) any three sets of prices ($p_1$,$p_2$ and $p_3$) in the three periods and (2) corresponding quantities for all commodities except one for one of the periods we can, by solving the equation of condition given above, find the value that the single remaining $q$ must have to satisfy the equation and, therefore, the condition that the circular test be satisfied in the special case. There are, clearly, any number of special cases for which the circular test will be satisfied. We are not, however, concerned with the question as to what may happen in these special cases but rather with the question, "Is it possible, assuming the $p$'s to take any sets of values assigned, to have the equation of condition reduce to an identity?" The answer is obvious: only if the quantities in the three years are the same ($q_1 = q_2 = q_3$). That is to say, the aggregative formula for a price index will satisfy the circular test (assuming the $p$'s to move as they will) only if the corresponding quantities are constant.

Likewise, the other formulas of our list of nine will not satisfy the circular test (i.e., that the quotient of *any* pair of index numbers to any base, $P_{rs}/P_{ts}$, equals the index number obtained for one of the pair of periods to the other of the pair as the base, $P_{rt}$) unless the weights are fixed.[1] If we assume fixed weights all of our

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[1] The condition that the geometric formulas i, ii and vi should satisfy the circular test for any three periods 1, 2 and 3 is

$$P_{31} = P_{21} \cdot P_{32} \text{ or}$$

$$\Pi \left(\frac{p_3}{p_1}\right)^{W_{31}} = \Pi \left(\frac{p_2}{p_1}\right)^{W_{21}} \left(\frac{p_3}{p_2}\right)^{W_{32}}$$

(where the $W$'s stand for different sets of value weights)

$$\text{or } \Pi p_3^{W_{31} - W_{21}} = \Pi p_2^{W_{21} - W_{32}} p_1^{W_{31} - W_{21}}$$

nine formulas reduce to two, (1) the geometric mean of relatives with fixed weights and (2) the aggregative mean with fixed quantities. These two and only these two forms of our averages, therefore, (1) are without weight correlation bias, (2) satisfy the base reversal test, and (3) satisfy the circular test.

If we use *any* fixed weights in the geometric average and *any* fixed quantities in the aggregative average the third condition named above is fulfilled. But we will not get good indexes of prices from these formulas *unless*, in the geometric average, the weights are based on prices and quantities which may justly be retained for all the periods in the interval for which index numbers are to be computed, and *unless*, in the aggregative average, the quantities may justly be retained for all the periods in the interval for which index numbers are to be computed.

For a binary comparison, the obvious fixed weights to use in the geometric mean are $\frac{1}{2}(p_1q/\Sigma p_1q+p_2q/\Sigma p_2q)$, which are the weights of formula i with fixed $q$'s where $q=$ the mean of $q_1$ and $q_2$, say $q=\frac{1}{2}(q_1+q_2)$. Likewise, for an aggregative mean, the obvious fixed quantities to use are $\frac{1}{2}(q_1+q_2)$. For a large number of

This will reduce to an identity for all values of $p_1$, $p_2$ and $p_3$ only if

$$W_{31}=W_{21}=W_{32}$$

That is to say, the geometric means will satisfy the circular test, in general, only if the weights are fixed and constant for the three periods.

periods ($n$) the obvious quantities to use are $1/n \cdot (q_1 + q_2 + \cdots q_n)$. The arithmetic mean of quantities gives a satisfactorily simple method of getting fixed weights for the aggregative formula, but in the geometric formula, the use of the fixed quantities with the price of each and every period is an unnecessarily laborious method of securing weights. We may substitute, therefore, in practice value weights obtained by combining (1) an average of prices for part or whole of the interval in question with (2) a corresponding average of quantities.

It will be observed from the preceding argument that the index numbers thus obtained (satisfying the circular test) will give useful results only for homogeneous intervals. This observation leads to the generalization that accurate comparisons are not possible over long intervals in which the $q$'s have changed radically. Further, even though we limit ourselves to a binary comparison of periods in which commodities and quantities are vastly different (say 1776 and 1928) the "ideal" formula would be of little aid in constructing an index of general prices. The difficulty is not in the lack of an average to use but in the data to be averaged.

That the circular test is the logical extension of the base reversal test follows from the fact that a necessary condition for the satisfaction of the circular test, constant weights, is also a necessary condition for the satisfaction of the base reversal test.

*Necessary condition for satisfaction of the base reversal test.* It will be noticed that formula v, the "ideal" form, is really a function of $2n$ relatives $\left(\frac{p_2}{p_1} \text{ and } \frac{p_1}{p_2} \text{ for } n \text{ commodities}\right)$. Formula v, with respect to period 1 as base, is

$$\left\{\frac{1}{\dfrac{\Sigma\left(\frac{p_1}{p_2}\right)p_2q_2}{\Sigma p_2q_2}}\cdot\frac{\Sigma\left(\frac{p_2}{p_1}\right)p_1q_1}{\Sigma p_1q_1}\right\}^{1/2}$$

This may be described as the square root of the product of

$$\left\{\begin{array}{c}\text{Reciprocal of the arith-}\\ \text{metic mean of relatives}\\ p_1/p_2 \text{ weighted by the}\\ \text{ratios } p_2q_2/\Sigma p_2q_2\end{array}\right\}\quad\left\{\begin{array}{c}\text{Arithmetic mean of}\\ \text{the relatives } p_2/p_1\\ \text{weighted by the ratios}\\ p_1q_1/\Sigma p_1q_1\end{array}\right\}$$

Formula v, with respect to period 2 as base, is

$$\left\{\frac{\Sigma\left(\frac{p_1}{p_2}\right)p_2q_2}{\Sigma p_2q_2}\cdot\frac{1}{\dfrac{\Sigma\left(\frac{p_2}{p_1}\right)p_1q_1}{\Sigma p_1q_1}}\right\}^{\frac{1}{2}}$$

This may be described as the square root of the product of

$$\left\{\begin{array}{c}\text{Arithmetic mean of}\\ \text{the relatives } p_1/p_2\\ \text{weighted by the ratios}\\ p_2q_2/\Sigma p_2q_2\end{array}\right\}\quad\left\{\begin{array}{c}\text{Reciprocal of the arith-}\\ \text{metic mean of the rela-}\\ \text{tive } p_2/p_1 \text{ weighted by}\\ \text{the ratios } p_1q_1/\Sigma p_1q_1\end{array}\right\}$$

The two expressions for the formula v obtained, respectively, by using each of two periods as the base both contain the same relatives with the same weights applied to those relatives. In other words, if we select the relative for any commodity and ascertain the weight of that relative in formula v, computed forward (with one of the two periods as the base), and then ascertain the weight of the same relative in formula v, computed backward (with the other of the two periods as the base), we shall find that the weights are identical. In still other words, the weight of the relatives with respect to each other and the weights of the reciprocals of the relatives with respect to each other are the same whether we compute the index forward or backward. The base reversal test is satisfied by formula v not because of changing emphasis (for the emphasis does not change) on the individual price relatives, but because the relatives that appear in the numerator of the index computed forward occur in the denominator of the index figured backward. The same generalization is true of formulas iii, iv and vii to ix inclusive.

It will also be noticed that in the geometric means of relatives, formulas i, ii and vi, the weights of the relatives computed forward are the same as the weights of the relatives computed backward, but the relatives appear in the numerator in one case and in the denominator in the other. Consequently, geometric means of relatives which satisfy the base reversal test and have no weight correlation bias, in fact have constant weights figured forward and backward.

In all of our nine formulas, therefore, the weights of the individual relatives for the indexes computed forward and backward are fixed. That is, even in a binary comparison the only averages of relatives that we have secured which (*a*) have no weight correlation bias and (*b*) satisfy the base reversal test are averages in which every relative has the same weight whether the first or the second period be taken as the base. To express the same idea in other words, in a binary comparison the weights of relatives actually computed for formulas i to ix inclusive with one period as the base are applied to precisely the same relatives when the other period is used as the base.

It appears further that, if any average is to satisfy the base reversal test a necessary condition is that the weight applied to each relative when one of the two periods is taken as the base must be the same as the weight applied to the same relative when the other of the two periods is taken as the base.

The obvious generalization of the base reversal test, which involves two periods only, is the circular test, which involves three or more periods. A necessary condition for satisfaction of the base reversal test is fixed weights when the index is computed forward and backward. Likewise a necessary condition for the satisfaction of the circular test is the use of the same fixed weights for all periods.

## III. COMPARISON OF FIXED WEIGHT INDEXES WITH VARIABLE WEIGHT INDEXES FOR THREE OR MORE PERIODS.

Professor Irving Fisher selected the formula which is numbered v in our system as "the best" or "the ideal" index number for an intercomparison of three or more periods as well as for a binary comparison. He selected the "ideal" index number from 13 formulas (these 13 are not all true index numbers, however) satisfying the base reversal and factor reversal tests[1] because the "ideal" was not only judged to be simpler in form than the others, but also because he had awarded it the first place in accuracy.[2] Taking the "ideal" formula as his standard of accuracy, Professor Fisher then ranked 119 formulas, on the basis of a comparison of the actual index numbers obtained for the years 1914–18 relative to 1913 for the special case of 36 series of commodity prices and their associated quantities.[3] Of the 119 formulas, the 13 ranking the highest — that is, most closely approximating the values of the "ideal" formula — according to Fisher, are the following:

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[1] *The Making of Index Numbers*, pp. 220–21.

[2] *Ibid.*, Table 28, pp. 244–47.

[3] *The Making of Index Numbers*, Appendix VI, pp. 489–90.

| *Ranking* | *Nature of Formulas* |
|---|---|
| 1, 2, 3 | (1) Geometric mean of $\Sigma p_2q_1/\Sigma p_1q_1$ and $\Sigma p_2q_2/\Sigma p_1q_2$, (2) arithmetic mean of same pair of aggregatives, and (3) factor antithesis of the last named formula, respectively. |
| 4, 6, 5 | (4) Arithmetically crossed weight aggregative $[\Sigma p_2(q_1+q_2)/\Sigma p_1(q_1+q_2)]$, (6) factor antithesis of the preceding formula, and (5) the geometric mean of the two formulas last named, respectively. |
| 9, 7, 8 | (9) Geometrically crossed weight aggregative $[\Sigma p_2\sqrt{q_1q_2}/\Sigma p_1\sqrt{q_1q_2}]$, (7) factor antithesis of the preceding formula, and (8) the geometric mean of the two formulas last named, respectively. |
| 13, 12, 10 | (13) Geometric average of relatives with geometric crossed weights, (12) factor antithesis of the preceding formula, and (10) the geometric mean of the two preceding formulas last named, respectively. |

| *Ranking* | *Nature of Formulas* |
|---|---|
| 11 | (11) Geometric mean of 8 geometric averages of relatives obtained by using each of the 4 sets of value weights and combining the index numbers, thus secured, with their factor antitheses. |

The 13 formulas listed above, which Fisher ranks highest with respect to his chosen standard, are not the same 13 which he had previously selected as satisfying the base reversal and factor reversal tests. In fact, the formulas ranking 2 and 3 satisfy neither test, and only those ranking 5, 8, 10 and 11 satisfy the factor reversal test. In fact, the 4 formulas last named were adjusted or "rectified" by Professor Fisher in order to make them satisfy the factor reversal test. These formulas are not, however, according to my notion, or according to Professor Fisher's definitions, true index numbers. Of the 13 ranking formulas listed above only those numbered 1, 4, 9 and 13 are true index numbers.

The index number used as the standard of comparison — and, therefore, automatically ranked first — is the "ideal."

The index number (not formula) ranked next is the arithmetic average of relatives with value weights obtained by using base year prices with the arithmetic means of quantities for the two years being compared. The index number ranked third is also an arithmetic average of relatives with value weights obtained by using base year prices, but these prices are combined with the geometric means (instead of arithmetic means) of quantities for the two years being compared.

The index number ranked fourth is the geometric average of relatives with value weights obtained by taking the geometric mean of the values in the two years being compared.

It is possible to generalize the index number ranked next to the standard (in the binary comparison) for an intercomparison of three or more periods. The constant value weights indicated in such a generalization would be obtained by using base year prices with the arithmetic mean of quantities for *all* the years being compared.

It is possible also to generalize, for a comparison of three or more periods, the index number ranked third to the standard. The constant weights that would be indicated in such a generalization would be obtained by using base year prices with the geometric means of quantities for all the years being compared. The only difference, it will be noticed, between the generalization thus indicated and that indicated in the previous paragraph is the difference between arithmetic and geometric means of the same set of quantities in the weights. Such a difference in weights would clearly result in a difference in the resulting index numbers of the nature of a higher order infinitesimal compared with the differences resulting from different methods of averaging and weighting the relatives themselves. Therefore, only one of the generalizations need be retained for actual use in computing index numbers. The generalization to be retained should be, naturally, the simpler of the two, that is, the one utilizing in obtaining weights the arithmetic, rather than the geometric, average of quantities.

It is possible to generalize, also, the index numbers ranked fourth to the standard for an intercomparison of three or more periods. The constant value weights indicated in such a generalization would be obtained by using the geometric mean of values for *all* the year being compared. It is evident, however, that the simpler arithmetic mean of values could be substituted in actual practice, for the geometric mean. Such a substitution, in actual practice, would result in a negligible difference.

We have, therefore, secured two index numbers generalized for an intercomparison of three or more periods, one an arithmetic average of relatives with an appropriate set of constant value weights and the other a geometric average of relatives with another appropriate set of constant value weights. These index numbers are the same as those derived on page 92 of Section II of the present article. The algebraic expression for these index numbers for the period $i$ relative to the period 1, respectively,

$$\Sigma p_i q / \Sigma p_1 q \text{ where } q = \frac{q_1 + q_2 + \cdots + q_n}{n}$$

$$\text{and } \Pi\left(\frac{p_i}{p_1}\right)^W \text{ where } W = pq/\Sigma pq$$

$$\text{and } p = \frac{p_1 + p_2 + \cdots + p_n}{n}$$

We may note in passing that these two index numbers computed for *each* of the five binary comparisons of prices for the years 1914–18 relative to 1913, using Professor Fisher's data for 36 commodities, give results remarkably close to those given by the "ideal" formula. The actual index numbers for prices thus computed are as follows:

TABLE 8.—BINARY COMPARISONS OF THREE INDEX NUMBERS OF PRICES, BASE: 1913*

| *Weighted averages* | *1914* | *1915* | *1916* | *1917* | *1918* |
|---|---|---|---|---|---|
| Ideal. . . . . . . . . . . . | 100.12 | 99.89 | 114.21 | 161.56 | 177.65 |
| Arithmetic†. . . . . . | 100.12 | 99.89 | 114.23 | 161.52 | 177.63 |
| Geometric. . . . . . . . | 100.1 | 99.9 | 113.8 | 162.1 | 177.8 |

* Data from *The Making of Index Numbers*, pp. 247.

† Aggregative mean. The weights used are indicated by the formula on p. 39.

We may also note in passing, first, that in a binary comparison the arithmetic average of relatives with weights obtained by using base year prices and $\frac{q_1+q_2}{2}$ will always lie between the two constituents of the "ideal" index and, second, that Professor Fisher has been unable to find any actual case in which the two index numbers differ by more than three-fourths of one per cent and, he remarks, "this, in most cases, would not be considered worth taking into account."[1]

Table 9 presents a comparison, for Fisher's special case of 36 commodities, of two index numbers of prices with fixed weights — the weighted geometric and arithmetic averages of relatives — with the "ideal" formula.[2] The maximum percentage deviation of the arithmetic mean with fixed weights from the "ideal" formula with variable weights is 0.33 of one per cent in 1914, and that of the geometric mean with fixed weights is 0.67 of one per cent in 1916.

Table 10 presents a similar comparison of

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[1] *The Making of Index Numbers*, Appendix I, p. 430.

[2] It is noteworthy that among the scores of index numbers which Fisher computed for his 36 commodities, he includes only one fixed weight formula, No. 9051, p. 518. This is a "round weight" formula obtained "after judicious shifts of decimal points of the 36 quotations." Even this round weight formula he classified as a "fair index number." If weights are more carefully chosen, fixed weight index numbers would have to be classified, using Fisher's terminology and standard of excellence, at least as "very good."[2]

TABLE 9. — COMPARISON OF INDEX NUMBERS OF PRICES FOR THE SPECIAL ILLUSTRATIVE CASE OF 36 COMMODITIES OF THREE AVERAGES WITH FIXED BASE AND WEIGHTS (*a*, *b*, *c*) WITH AVERAGES WITH VARIABLE WEIGHTS (*d*, *e*, *f*).
(*Base: 1913*)

| | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|
| (*a*) Formula 9051 (Fisher)[1] | 101.68 | 103.10 | 113.63 | 160.37 | 182.07 |
| (*b*) Geometric mean with fixed weights[2] | 99.52 | 99.55 | 114.97 | 160.53 | 178.42 |
| (*c*) Arithmetic mean with fixed weights[3] | 99.79 | 99.85 | 114.04 | 161.61 | 177.88 |
| (*d*) Ideal (formula v) | 100.12 | 99.89 | 114.21 | 161.56 | 177.65 |
| (*e*) Range of Fisher's "excellent" and "superlative" indexes | 100.09<br>to<br>100.16 | 99.84<br>to<br>99.96 | 113.99<br>to<br>114.35 | 161.18<br>to<br>161.94 | 177.34<br>to<br>178.16 |
| (*f*) Range of the index numbers of formulas (i) to (xi) | 100.09<br>to<br>100.12 | 99.98<br>to<br>100.06 | 113.83<br>to<br>114.59 | 161.37<br>to<br>163.10 | 176.63<br>to<br>177.80 |

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[1] This is a "round weight" aggregative formula in which the weighting is obtained "after judicious shifts of decimal points of the 36 quotations" (Fisher, *The Making of Index Numbers*, p. 465).

[2] The weights of the relatives are the ratios $pq/\Sigma pq$ where $p$ is the arithmetic mean of prices and $q$ is the arithmetic mean of quantities for the six years, 1913–18.

[3] The weights of the relatives are the ratios $p_{1913}q/\Sigma p_{1913}q$, where $p_{1913}$ is the price in 1913 and $q$ is the arithmetic mean of quantities for 1913–18.

index numbers of quantities. The maximum percentage differences between the arithmetic and geometric means with fixed weights on the one hand, and the "ideal" formula on the other, are, respectively, 0.38 of one per cent in 1914 and 0.56 of one per cent in 1916.

Tables 11 and 12 present comparisons of two index numbers, arithmetic and geometric means, with fixed weights for Fisher's special case of 36 commodities, the index numbers, however, being computed for three sub-groups as well as for all commodities. The maximum absolute divergence of corresponding indexes for quantities is in 1916 and for the subgroup of "15 commodities with weights of less than one per cent each." In this instance, the weighted geometric mean is 121.23 as compared with 116.64 for the weighted arithmetic mean, a divergence of about 4 per cent. In 1914 and 1915 the absolute divergence was nearly as great as in 1916.

Table 13 presents comparisons of the value ratios for the 36 commodities and each subgroup, on the one hand, with the products of corresponding indexes of prices and quantities, on the other hand. This comparison shows the extent of the deviation of the products of corresponding fixed weight indexes for prices and quantities from perfect satisfaction of the factor reversal test. For the 36 commodities the maximum percentage deviation of the product of corresponding arithmetic indexes for prices and quantities is 0.71

Table 10. — Comparison of Index Numbers of Quantities for the Special Illustrative Case of 36 Commodities of Two Averages with Fixed Base and Weights (*b*, *c*) with (*d*) the Ideal Index (Formula v).

(*Base: 1913*)

| | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|
| (*a*) Formula 9051 (Fisher)[1] | .. | .. | .. | .. | .. |
| (*b*) Geometric mean with fixed weights[2] | 99.68 | 109.11 | 119.52 | 119.14 | 124.89 |
| (*c*) Arithmetic mean with fixed weights[3] | 98.95 | 108.73 | 118.94 | 118.98 | 125.26 |
| (*d*) Ideal (formula v) | 99.33 | 109.10 | 118.85 | 118.98 | 125.37 |

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[1] Fisher does not give the index numbers of quantities corresponding to formula 9051 for prices. The present writer is uncertain how to make "judicious shifts of decimal points" of the 36 quantities in order to get a quantity index correlative to Fisher's price index 9051.

[2] The weights of the relatives are the ratios $pq/\Sigma pq$ where $p$ is the arithmetic mean of prices and $q$ is the arithmetic mean of quantities for the six years, 1913–18.

[3] The weights of the relatives are the ratios $pq_{1913}/\Sigma pq_{1913}$, where $p$ is the arithmetic mean of prices for 1913–18 and $q_{1913}$ is the quantity in 1913.

Table 11. — Comparison of Index Numbers of Prices of Each of Three Subgroups of 36 Commodities, Computed by Using (*a*) the Geometric Mean with Fixed Base and Weights and (*b*) the Arithmetic Mean with Fixed Base and Weights.

(*Base: 1913*)

| Groups* | Aggregate Weights* | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|---|
| 11 Commodities with weights of 4% or over... | 72.39 | | | | | |
| (*a*) Geometric mean.................... | | 101.77 | 100.83 | 111.62 | 155.20 | 173.03 |
| (*b*) Arithmetic mean.................... | | 101.89 | 101.19 | 110.74 | 156.77 | 172.30 |
| 10 Commodities with weights of 1% and less than 4%........................ | 20.32 | | | | | |
| (*a*) Geometric mean.................... | | 94.72 | 99.52 | 128.62 | 189.05 | 208.21 |
| (*b*) Arithmetic mean.................... | | 95.44 | 100.28 | 128.64 | 189.91 | 209.19 |
| 15 Commodities with weights of less than 1%. | 7.26 | | | | | |
| (*a*) Geometric mean.................... | | 91.50 | 87.64 | 112.81 | 142.20 | 157.19 |
| (*b*) Arithmetic mean.................... | | 90.38 | 86.37 | 110.79 | 140.77 | 156.84 |
| 36 Commodities............................ | 99.97 | | | | | |
| (*a*) Geometric mean.................... | | 99.52 | 99.55 | 114.97 | 160.53 | 178.42 |
| (*b*) Arithmetic mean.................... | | 99.79 | 99.85 | 114.04 | 161.61 | 177.88 |

*Grouping of the commodities is based on the weights used in the geometric mean.

Table 12.— Comparison of Index Numbers of Quantities of Each of Three Subgroups of 36 Commodities, Computed by Using (*a*) the Geometric Mean with Fixed Base and Weights and (*b*) the Arithmetic Mean with Fixed Base and Weights.

(*Base: 1913*)

| Groups* | Aggregate Weights* | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|---|
| 11 Commodities with weights of 4% or over... | 72.39 | | | | | |
| (*a*) Geometric mean.................. | | 98.58 | 106.12 | 116.85 | 116.11 | 123.94 |
| (*b*) Arithmetic mean.................. | | 98.68 | 106.02 | 116.72 | 116.33 | 124.65 |
| 10 Commodities with weights of 1% and less than 4%........................ | 20.32 | | | | | |
| (*a*) Geometric mean.................. | | 99.76 | 119.66 | 128.87 | 128.10 | 130.87 |
| (*b*) Arithmetic mean.................. | | 100.77 | 119.59 | 128.29 | 127.41 | 130.95 |
| 15 Commodities with weights of less than 1%. | 7.26 | | | | | |
| (*a*) Geometric mean.................. | | 101.05 | 111.14 | 121.23 | 125.70 | 118.31 |
| (*b*) Arithmetic mean.................. | | 96.92 | 107.30 | 116.64 | 123.32 | 116.45 |
| 36 Commodities........................ | 99.97 | | | | | |
| (*a*) Geometric mean.................. | | 99.68 | 109.11 | 119.52 | 119.14 | 124.89 |
| (*b*) Arithmetic mean.................. | | 98.95 | 108.73 | 118.94 | 118.98 | 125.26 |

*Grouping of the commodities is based on the weights used in the geometric mean.

TABLE 13. — COMPARISON WITH VALUE RATIOS OF PRODUCTS OF INDEX NUMBERS OF PRICES AND QUANTITIES OF EACH OF THREE SUBGROUPS OF 36 COMMODITIES, COMPUTED BY USING (*a*) THE GEOMETRIC MEAN WITH FIXED BASE AND WEIGHTS AND (*b*) THE ARITHMETIC MEAN WITH FIXED BASE AND WEIGHTS.

(*Base: 1913*)

| Groups* | Aggregate Weights* | 1914 | 1915 | 1916 | 1917 | 1918 |
|---|---|---|---|---|---|---|
| 11 Commodities with weights of 4% or over | 72.39 | | | | | |
| (*a*) $G^{(p)} \cdot G^{(q)}$ | | 100.32 | 107.00 | 130.43 | 180.20 | 214.45 |
| (*b*) $A^{(p)} \cdot A^{(q)}$ | | 100.55 | 107.28 | 129.26 | 182.37 | 214.77 |
| (*c*) Value ratios | | 100.95 | 107.68 | 129.07 | 181.91 | 214.70 |
| 10 Commodities with weights of 1% and less than 4% | 20.32 | | | | | |
| (*a*) $G^{(p)} \cdot G^{(q)}$ | | 94.49 | 119.09 | 165.75 | 242.17 | 272.48 |
| (*b*) $A^{(p)} \cdot A^{(q)}$ | | 96.17 | 119.92 | 165.03 | 241.96 | 273.93 |
| (*c*) Value ratios | | 97.37 | 120.24 | 164.95 | 241.97 | 272.97 |
| 15 Commodities with weights of less than 1% | 7.62 | | | | | |
| (*a*) $G^{(p)} \cdot G^{(q)}$ | | 92.46 | 97.40 | 136.76 | 178.75 | 185.97 |
| (*b*) $A^{(p)} \cdot A^{(q)}$ | | 87.60 | 92.68 | 129.23 | 173.60 | 182.64 |
| (*c*) Value ratios | | 89.84 | 95.86 | 133.57 | 178.35 | 185.54 |
| 36 Commodities | 99.97 | | | | | |
| (*a*) $G^{(p)} \cdot G^{(q)}$ | | 99.20 | 108.62 | 137.41 | 191.26 | 222.83 |
| (*b*) $A^{(p)} \cdot A^{(q)}$ | | 98.74 | 108.57 | 135.64 | 192.28 | 222.81 |
| (*c*) Value ratios | | 99.45 | 108.98 | 135.75 | 192.23 | 222.71 |

*Grouping of the commodities is based on the weights used in the geometric mean.

TABLE 14.—DATA FOR THE INDEXES OF QUANTITY FOR THE SUBGROUP OF 15 COMMODITIES FOR THE YEAR 1916.*

| Commodity | Relatives $q_{1916}/q_{1913}$ | Aggregative weights $\frac{p_{1913-18} \cdot q_{1913}}{\Sigma p_{1913-18} \cdot q_{1913}}$ | Geometric weights $\frac{p_{1913-18} \cdot q_{1913-18}}{\Sigma p_{1913-18} \cdot q_{1913-18}}$ | Effect of weights on index |
|---|---|---|---|---|
| Rubber | 224 | .45 | .86 | Geom. higher |
| Tin plate | 149 | .50 | .65 | Geom. higher |
| Coffee | 139 | .48 | .55 | Geom. higher |
| Tin, pig | 138 | .33 | .37 | Geom. higher |
| Lead | 134 | .30 | .34 | Geom. higher |
| Silk | 128 | .59 | .64 | Geom. higher |
| Lime | 116 | .22 | .20 | Geom. higher |
| Cement | 107 | .97 | .85 | Geom. higher |
| Lard | 104 | 1.03 | .88 | Geom. higher |
| Coal, anthracite | 98 | .20 | .21 | Geom. lower |
| Silver | 95 | .61 | .54 | Geom. higher |
| Lead, white | 90 | .16 | .13 | Geom. higher |
| Mutton | 84 | .60 | .45 | Geom. higher |
| Skins | 84 | .16 | .09 | Geom. higher |
| Steel rails | 82 | .74 | .50 | Geom. higher |

*The weighted geometric mean is 121.23 as compared with 116.64 for the weighted arithmetic mean.

of one per cent in 1914 — in all other years the deviation is less than one-half of one per cent. For the same group of commodities the maximum percentage deviation of the products of correlative geometric indexes for prices and quantities is 1.22 per cent in 1916. Among the subgroups there is an especially marked divergence in the subgroup of 15 commodities. In this subgroup the divergence is most pronounced in 1916.

The divergence of the geometric and arithmetic indexes for the subgroups for 15 commodities is so substantial and so much greater than in other subgroups, as shown by Tables 11, 12, and 13 that a special investigation of the source of these discrepancies was undertaken. Since the discrepancies appeared more emphatically in the indexes for quantities than for prices, attention was directed to the quantity relatives. The data for the indexes of quantities for the subgroup of 15 commodities for the year 1916 are given in Table 14.

Table 14 reveals a remarkable situation among the quantity relatives and their weights in the subgroup for 15 commodities. It happens that for every one of the larger relatives, *i.e.*, over 121.23, the weights are *heavier* for the geometric mean than for the arithmetic mean and for all of the small relatives, *i.e.*, under 116.64, excepting that for anthracite coal, only, the weights are *lighter* for the geometric mean than for the arithmetic mean. The large discrepancy between the two means, therefore, arises from the accidental positive correlation between the weights and relatives of the geometric mean as compared with that between the weights and relatives of the arithmetic mean. If we use the geometric weights in the arithmetic mean of relatives we get an index number of 117 (instead of 116.64) and if we use the arithmetic weights in the geometric mean of relatives we get an index number of 106 (instead of 121.23). It is obviously the weights rather than the method of averaging that produces the discrepancy in this instance.

## IV. Special Difficulties (Other than the Problem of Averaging Given Homogeneous Data) of Obtaining Satisfactory Price Indexes for Two or More Periods or Places

In our discussion up to this point we assumed that we had at our disposal data for prices and corresponding quantities for a number ($n$) of commodities or articles in one period (1) and for precisely the same commodities or articles for another period (2). In assuming that there were $n$ commodities in both periods we implicitly assumed also that we had $n$ finite price relatives, *i.e.*, that none of the $p$'s were zero or infinite, and that we had $n$ finite quantities, *i.e.*, that none of the $q$'s were zero or infinite. That is to say, we excluded free goods and goods not common to both periods. Upon the basis of these assumptions we found satisfactory averages of prices to use either for a binary comparison or for a comparison of two or more periods (the aggregative formula with fixed quantities obtained by taking the aggregates of the quantities for all the periods being compared).

In any actual problem of finding the average change of prices between any two years we must, at the outset, assemble the data. The difficulties encountered in this process are:

(1) *Definition of the field.* Is it our object to measure the average change in prices of material commodities only, such as wheat, hides, potatoes, automobiles, vacuum sweepers, chairs and houses, or of material commodities and services, such as electrical power for light and refrigeration, taxi rides, haircuts, voyages to Europe, seats at the theatre, rooms at hotels, and wages of servants, or of material commodities, services and investments, such as stocks, bonds, houses and land? Do we want to limit the field to certain varieties of economic goods, such as building materials or food, or certain classes of prices, wholesale or retail, or certain quantities of commodities, such as to exclude commodities with nominal quantities in either period? Do we want to include raw materials, partly manufactured goods and finished goods? Shall we include by-products, patented articles, and monopolistic goods?

Shall we include or exclude the long list of articles, like fabrics, shoes and mah-jong sets in which the style element is a leading factor in price? Do we want to have reflected in the "average change of price" (which we are attempting to get) changes in supply and demand connected with the individual commodities, regardless of the nature of the changes, and changes on the money side, regardless of their nature? In general the definition of the field is determined by the purpose of the index number to be constructed.

(2) *It is difficult to get $p$'s and $q$'s that exactly correspond.* There may be, for instance, five or ten different grades of a commodity, such as wheat. If we take only one or two series of price quotations we must face the question: "Should the corresponding $q$ be the quantity of that grade of wheat or of all wheat sold?" The answer to this question depends upon the purpose of our index. Often we are obliged to use quantities that correspond exactly neither to our individual price series nor the total of the commodity represented by our individual price series. Further, we may not be able to find $q$'s to match prices for every time unit (week, month, quarter, or year) for which we desire price indexes. Even though we desire to use variable weights we can not get them, usually, for as frequent intervals as we can get prices.

(3) *Procedure if $q_1$ or $q_2$ equals zero.* Suppose

that our object has been completely defined and it is to secure a comprehensive index of the cost of living in 1928 as compared with that in 1870. We find that in 1870 lighting was by kerosene lamps and that kerosene was a significant item in the budget whereas in 1928 lighting is by electricity. Electric power was not sold in 1870 so that for this element in the budget $q_1 = 0$. We are therefore obliged to omit electric power as such. But we might eliminate both kerosene and electricity from our list of price series and substitute "domestic illumination" in lieu of them. If we do this we would have to determine upon homogeneous units of domestic illumination in the two periods, and ascertain the number and prices of those units in 1870 and 1928. If this is impossible or inexpedient we must be satisfied with an index number which reflects only part of the changes in the cost of living. Excluding electricity we ought also to exclude kerosene from our list of elements in the cost of living.

(4) *Changes in the nature of the commodity with time.* A similar problem occurs when we are comparing, say, the cost of food in July with that in January. The commodity, potatoes, for instance, means mainly old potatoes in January and new potatoes in July. Instead of one series, we must have two, and if either series disappears entirely in one or the other period ($q_1$ or $q_2 = 0$) the commodity must be eliminated from our list. Likewise, we may find it necessary to subdivide

other seasonal commodities into several homogeneous commodities, such as eggs into the classes depending upon the age of the commodity. There are also other commodities than foods, such as "packer's heavy native steer hides," which have a seasonal variation in price because the commodity changes with the seasons. Thus, steer hides of a fixed grade are cheaper in the spring than in the autumn because the hides are apt to have blemishes acquired during the winter season which are not acquired to the same extent in summer. In this case it is logical to make some adjustment for the seasonal change of price if the commodity is to be retained in our index.

(5) *Special difficulties of obtaining satisfactory price indexes for a series of periods.* The difficulties of obtaining homogeneous price series for two periods are, of course, even more apt to be encountered if our object is to get a continuous series of price indexes. Having set up a list of satisfactory price series and having secured satisfactory weights for one interval we may find that another or an amended list of prices and weights is more satisfactory for an ensuing interval. We may be obliged to compute our index numbers for separate intervals and splice the index numbers in some fashion to get a continuous series. Or, in constructing a current extension of a series of index numbers we may suddenly find that one or more price series are no longer pro-

curable. In this case, unless we recompute the index from the beginning, we are obliged to make some sort of substitution of a new price series for the old.

(6) *Chain indexes.* In this connection we may say that the chain method of computing index numbers is adapted to making changes in commodity lists. It has been proven, however, to be less reliable than the fixed base method as a means of obtaining a homogeneous series of price indexes. The relation between chain and fixed base indexes will be discussed in the following section.

## V. THE DIVERGENCE OF CHAIN INDEXES FROM CORRESPONDING FIXED BASE INDEXES

If fixed weights are used for all years, any average which satisfies the base reversal test will also satisfy the circular test. With fixed weights, therefore, the chain index for any year, computed by an average which satisfies the base reversal test, will be identical with the fixed base index for the same year computed by the same formula.

If, however, the weights depend upon the data of the pairs of years for which the links of the chain index are computed, the resulting chain index and the corresponding fixed base index for the same year, computed by the same formula, will, in general, differ.

It has frequently been observed, in actual practice, not only that chain indexes with changing weights differ from corresponding fixed base indexes, but also that the divergence of the two indexes tends to persist and even to become irregularly greater and greater as the number of links in the chain increases. Thus, in the case of Fisher's 36 commodities the chain indexes, computed by our nine formulas as shown in Table 6, are greater than the corresponding fixed base indexes in every instance. In a second case, *i.e.*, the indexes of the prices and quantities of twelve leading crops, 1910–19, computed by formula v by the chain method and fixed base method, the chain index for prices is always less than the corresponding fixed base index for prices and the chain index for quantities is always greater than the corresponding fixed base index for quantities.[1]

In spite of the divergence of chain indexes from fixed base indexes some students of the subject have recommended the use of the former, but no one has made a thoroughgoing investigation of the causes of the divergence.[1]

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[1] It was, apparently, on the basis of results actually found in special cases that led Fisher to say, "the chain is subject to cumulative error and ought not to be used (unless, possibly, as supplementary to the fixed base system)." But he inconsistently says on another page, "the present investigation has shown that, in the case of all good index numbers, there is no really perceptible difference between the chain and fixed base figures." See *The Making of Index Numbers*, p. 319 and p. 309.

For the indexes of prices and quantities of twelve crops see "Fisher's Formula for Index Numbers," by Warren M. Persons, in the *Review of Economic Statistics*, May 1921, p. 110.

Some have ascribed the divergence in a single year to chance and the cumulative divergence over a period of years to a "run of luck." Such an explanation, however, does not carry conviction, because prices and quantities in consecutive years are not "independent events" subject to the laws of games of chance.

As a basis for discussing the cause of the divergence between chain indexes and fixed base indexes computed by corresponding formulas, we shall find expressions for the rates of divergence between them. We shall derive and consider such expressions for (*a*) formula v, the "ideal," and (*b*) formula i, the geometric mean.

(*a*) Let the ratio of the chain index to the fixed base index (computed by formula v in both cases) for year $n$ be represented by $R_n$. Then

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[1] Thus, C. M. Walsh in the *Problem of Estimation*, 1921, pp. 105, 137, says that a Committee of the British Association of which Professor F. G. Edgeworth was secretary, "rightly recommended what has since been called the 'chain' system of making index numbers for every year compared with its predecessor."

The discussion of chain indexes by Wesley C. Mitchell in the *Bulletin of the U. S. Bureau of Labor Statistics*, No. 284, pp. 81–91, is useless for our purpose because it is limited to a comparison of chain and fixed base indexes computed by taking the simple arithmetic mean of relatives. The divergence between the two indexes thus obtained results mainly from the facts that the arithmetic average has an upward base bias and that this bias is cumulative in the chain index.

$$R_2 = \frac{P_{2\,1}}{P_{2\,1}} = 1$$

$$R_3 = \frac{P_{3\,2} \cdot P_{2\,1}}{P_{3\,1}} = \left(\frac{\Sigma p_3 q_2}{\Sigma p_2 q_3} \cdot \frac{\Sigma p_2 q_1}{\Sigma p_1 q_2} \cdot \frac{\Sigma p_1 q_3}{\Sigma p_3 q_1}\right)^{\frac{1}{2}}$$

$$R_4 = \left(\frac{\Sigma p_4 q_3}{\Sigma p_3 q_4} \cdot \frac{\Sigma p_3 q_2}{\Sigma p_2 q_3} \cdot \frac{\Sigma p_2 q_1}{\Sigma p_1 q_2} \cdot \frac{\Sigma p_1 q_4}{\Sigma p_4 q_1}\right)^{\frac{1}{2}}$$

. . . . . . . . . . .

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$$R_n = \left(\frac{\Sigma p_n q_{n-1}}{\Sigma p_{n-1} q_n} \cdot \frac{\Sigma p_{n-1} q_{n-2}}{\Sigma p_{n-2} q_{n-1}} \cdot \ldots \frac{\Sigma p_2 q_1}{\Sigma p_1 q_2} \cdot \frac{\Sigma p_1 q_n}{\Sigma p_n q_1}\right)^{\frac{1}{2}}$$

$$R_{n+1} = \left(\frac{\Sigma p_{n+1} q_n}{\Sigma p_n q_{n+1}} \cdot \frac{\Sigma p_n q_{n-1}}{\Sigma p_{n-1} q_n} \ldots \frac{\Sigma p_2 q_1}{\Sigma p_1 q_2} \cdot \frac{\Sigma p_1 q_{n+1}}{\Sigma p_{n+1} q_1}\right)^{\frac{1}{2}}$$

Dividing $R_{n+1}$ ($n+1$ factors) by $R_n$($n$ factors) we have:

$$\frac{R_{n+1}}{R_n} = \left(\frac{\Sigma p_{n+1} q_n}{\Sigma p_n q_{n+1}} \cdot \frac{\Sigma p_1 q_{n+1}}{\Sigma p_{n+1} q_1} \cdot \frac{\Sigma p_n q_1}{\Sigma p_1 q_n}\right)^{\frac{1}{2}}$$

which is the expression for the rate of change of the chain index as compared with the fixed base index between any two consecutive periods. This ratio will be greater than, equal to, or less than 1. That is to say:

$\frac{R_{n+1}}{R_n}$ is either $\gtreqless 1$, for any given values of $p$ and $q$.

The relationship just given may be expressed in terms of three weighted averages of relative prices as follows:

$$\frac{\Sigma\frac{p_{n+1}}{p_1}(p_1q_n)}{\Sigma p_1q_n}\Bigg/\frac{\Sigma\frac{p_n}{p_1}(p_1q_{n+1})}{\Sigma p_1q_{n+1}} \gtreqless \frac{\Sigma\frac{p_{n+1}}{p_n}(p_nq_1)}{\Sigma p_nq_1}$$

There are, of course, any number of special cases in which $\frac{R_{n+1}}{R_n}=1$. In such special cases the chain and fixed base indexes will coincide or move parallel. In general, however, the chain and fixed base indexes will diverge. Not only will the two indexes usually diverge, but they will diverge, in general, even though prices of all commodities in period $(n+1)$ are precisely the same as in period $(n)$.

We may illustrate the sort of thing that may occur in actual practice by taking an extreme case of the average of the prices of two commodities with fixed prices in all periods except the base period, but with quantities which vary in pronounced fashion from period to period. Consider commodities *A* and *B* for 7 periods as follows:

| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Commodity A: | $p$ | 2 | 10 | 10 | 10 | 10 | 10 | 10 |
| | $q$ | 100 | 50 | 1 | 50 | 100 | 50 | 50 |
| Commodity B: | $p$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | $q$ | 1 | 50 | 100 | 50 | 1 | 50 | 50 |

The chain and fixed base indexes computed by formula v are as follows:

| | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| Chain index...... | 427 | 427 | 427 | 427 | 427 | 427 |
| Fixed base index.. | 427 | 229 | 427 | 498 | 427 | 427 |

It will be noticed that the chain index remains fixed at 427 while the fixed base index fluctuates widely, between 229 and 498. This illustration shows that by making the quantities vary in a certain fashion from year to year it is possible to make the fixed base index change more or less rapidly (as we wish) than the chain index.

Consider, in this connection, the indexes for the periods 4 and 5 in which the fixed base index moves upward faster than the chain index, *i.e.*, the chain index diverges downward. It is obvious that the chain index would still tend to diverge downward if the prices of commodities *A* and *B* did not remain absolutely fixed, but instead merely maintained their approximate positions relative to the fixed base. That is to say, if prices do not remain absolutely fixed from period to period, but change slightly so that high prices in one period are "about as high" in the succeeding period and low prices in one period are "about as low" in the succeeding period the results of the illustration just considered will tend to be approximated by the chain and fixed base indexes.[1]

The illustration which we have just considered suggests that, in actual practice, if important commodities rise in price more than the average (relative to a fixed base) and remain high in price in the succeeding period (relative to the average for that period), regardless of higher production, and if important commodities fall in price more than the average (relative to a fixed base) and remain low in price regardless of smaller production (*i.e.*, if in general it is difficult to adjust production to prices and if more than one year is required for adjustment and in the meantime production is increased due to stimulus of high prices but not enough to cause a significant decline in price, and production of other commodities is curtailed on account of lower prices but not enough to cause a significant rise in prices) then

$$\frac{\Sigma \frac{p_{n+1}}{p_1}(p_1 q_n)}{\Sigma p_1 q_n}$$

will tend to be smaller, on account of the weights $q_n$, than

$$\frac{\Sigma \frac{p_n}{p_1}(p_1 q_{n+1})}{\Sigma p_1 q_{n+1}}$$

which will tend to be larger, on account of the weights $q_{n+1}$

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[1] The effect of individual weights and relatives upon the resulting index number should be judged not with respect to 100, *i.e.*, whether individual prices are increasing or decreasing, but with respect to $P$, *i.e.*, whether individual prices are changing more or less rapidly than the average.

and

$$\frac{\Sigma\frac{p_{n+1}}{p_n}(p_nq_1)}{\Sigma p_nq_1}$$

will tend to equal 1 on account of the approximate equality of $p_{n+1}$ and $p_n$

and

$$\frac{R_{n+1}}{R_n}$$

will tend to $<1$

and

the fixed base index will tend upward faster than the chain index or, in other words, the chain index will diverge downward.

The situation just described was actually found to exist for the chain and fixed base index numbers of prices for 12 leading crops during the years of pronounced divergence of the two indexes in 1914–15, 1916–17, and 1918–19.[1] The chain and fixed base indexes of prices of 12 crops for the years 1910–19 are as follows:

| | *1910* | *1911* | *1912* | *1913* | *1914* | *1915* | *1916* | *1917* | *1918* | *1919* |
|---|---|---|---|---|---|---|---|---|---|---|
| Chain Index.. | 100.0 | 109.2 | 93.7 | 110.8 | 100.6 | 100.7 | 152.0 | 206.4 | 217.4 | 229.7 |
| Fixed Base Index...... | 100.0 | 109.2 | 94.1 | 110.9 | 100.8 | 102.3 | 154.0 | 211.7 | 222.2 | 239.5 |

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[1] See "Fisher's Formula for Index Numbers," by Warren M. Persons, in the *Review of Economic Statistics*, May 1921, p. 110.

In each of the intervals 1914–15, 1916–17, 1918–19, the prices of important (*i.e.*, heavily weighted) commodities that were relatively high (with respect to 1910) in the first year of the interval remained relatively high in the second year of the interval and the prices of important commodities that were relatively low in the first year remained relatively low in the second year of the interval. Also $q_{n+1}$ in general was greater than $q_n$ for the relatively high-priced commodities and $q_{n+1}$ was, in general, less than $q_n$ for the low-priced commodities.

We may further add, in confirmation of the generalization, that an actual case of the converse relationship (to that for the prices of 12 crops) between the chain and fixed base indexes is afforded by Fisher's 36 commodities. From Table 6 we see that the greatest divergence of the chain and fixed base indexes was in the interval 1914–15. In 1914 the "ideal" indexes are coincident but in 1915 the fixed base index was only 99.89 compared with 100.23 for the chain index. That is to say, the chain index diverged upward (instead of downward as in the case of the prices of 12 crops for 1914–15). Investigation of the position of the price relatives of the most important commodities in 1914 and 1915 showed that the high relatives in 1914 did *not* remain high in 1915. Thus, pork, one of the most important of Fisher's 36 commodities, shifted its position from 4 per cent above the average to 4 per cent below the average. Thus, a condition that was approximately fulfilled for the relative prices of 12 crops in 1914–15 failed to be fulfilled for important constituents of Fisher's 36 commodities for the same interval. We find, in fact, that the chain index for the 36 commodities diverged upward in 1914–15 instead of downward as it did for the prices of 12 crops.

We may prove, further, that if the chain index for prices (computed by formula v, the "ideal") is greater than the fixed base index for prices, computed by the same formula, then the correlative chain index for quantities will be less than the fixed base index for quantities. The proof follows:

Let $R_n^{(p)}$ and $R_n^{(q)}$ be the ratios of the chain indexes to the fixed base indexes of prices and quantities, respectively. Then

$$R_n^{(p)}=\left(\frac{\Sigma p_nq_{n-1}}{\Sigma p_{n-1}q_n}\cdot\cdot\cdot\cdot\frac{\Sigma p_2q_1}{\Sigma p_1q_2}\cdot\frac{\Sigma p_1q_n}{\Sigma p_nq_1}\right)^{\frac{1}{2}}$$

$$R_n^{(q)}=\left(\frac{\Sigma q_np_{n-1}}{\Sigma q_{n-1}p_n}\cdot\cdot\cdot\cdot\frac{\Sigma q_2p_1}{\Sigma q_1p_2}\cdot\frac{\Sigma q_1p_n}{\Sigma q_np_1}\right)^{\frac{1}{2}}$$

and $R_n^{(p)}\cdot R_n^{(q)}=1.$

Since $R_n^{(p)}\cdot R_n^{(q)}=1=$

$$\frac{\text{(chain index of prices) (chain index of quantities)}}{\text{(fixed base index of prices) (fixed base index of quantities)}}$$

and the denominator of the right-hand side of the equation equals $\Sigma p_nq_n/\Sigma p_1q_1$ or the value ratio, the numerator will also equal the value ratio. The theorem stated above immediately follows.

(*b*) The expression for $\frac{R_{n+1}}{R_n}$ depends, of course, upon the formula used for the chain and fixed base average. It is hardly worth while to find the expression for all of our nine formulas, but we shall find it for one other average, formula i. The rate of change of the chain index relative to the fixed base index for any interval is:

$$\frac{R_{n+1}}{R_n}=\frac{\Pi\left(\frac{p_{n+1}}{p_n}\right)^{W_{n+1\cdot n}}\left(\frac{p_n}{p_1}\right)^{W_{n\cdot 1}}}{\Pi\left(\frac{p_{n+1}}{p_1}\right)^{W_{n+1\cdot 1}}}$$

where Π stands for "the product of such terms as" and

$$W_{r\cdot s}=\tfrac{1}{2}\left(\frac{p_rq_r}{\Sigma p_rq_r}+\frac{p_sq_s}{\Sigma p_sq_s}\right)$$

The expression for the rate of change for the chain index relative to the fixed base index, using geometric means, is derived as follows:

$$R_2=\frac{\Pi\left(\frac{p_2}{p_1}\right)^{W_{21}}}{\Pi\left(\frac{p_2}{p_1}\right)^{W_{21}}}=1$$

$$R_3 = \frac{\Pi\left(\frac{p_2}{p_1}\right)^{W_{21}}\left(\frac{p_3}{p_2}\right)^{W_{32}}}{\Pi\left(\frac{p_3}{p_1}\right)^{W_{31}}}$$

$$R_4 = \frac{\Pi\left(\frac{p_2}{p_1}\right)^{W_{21}}\left(\frac{p_3}{p_2}\right)^{W_{32}}\left(\frac{p_4}{p_3}\right)^{W_{43}}}{\Pi\left(\frac{p_4}{p_1}\right)^{W_{41}}}$$

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$$R_n = \frac{\Pi\left(\frac{p_2}{p_1}\right)^{W_{21}}\left(\frac{p_3}{p_2}\right)^{W_{32}}\cdots\left(\frac{p_n}{p_{n-1}}\right)^{W_{n\cdot n-1}}}{\Pi\left(\frac{p_n}{p_1}\right)^{W_{n\cdot 1}}}$$

$$R_{n+1} =$$

$$\frac{\Pi\left(\frac{p_2}{p_1}\right)^{W_{21}}\left(\frac{p_3}{p_2}\right)^{W_{32}}\cdots\left(\frac{p_n}{p_{n-1}}\right)^{W_{n\cdot n-1}}\left(\frac{p_{n+1}}{p_n}\right)^{W_{n+1\cdot n}}}{\Pi\left(\frac{p_{n+1}}{p_1}\right)^{W_{n+1\cdot 1}}}$$

$$\frac{R_{n+1}}{R_n} = \frac{\Pi\left(\frac{p_{n+1}}{p_n}\right)^{W_{n+1\cdot n}}\left(\frac{p_n}{p_1}\right)^{W_{n\cdot 1}}}{\Pi\left(\frac{p_{n+1}}{p_1}\right)^{W_{n+1\cdot 1}}}$$

Where $p_{n+1} = p_n$ we have

$$\frac{R_{n+1}}{R_n} = \frac{\Pi\left(\frac{p_n}{p_1}\right)^{W_{n\cdot 1}}}{\Pi\left(\frac{p_n}{p_1}\right)^{W_{n+1\cdot 1}}} = \Pi\left(\frac{p_n}{p_1}\right)^{W_{n\cdot 1} - W_{n+1\cdot 1}}$$

in which $W_{n\cdot 1} = \frac{1}{2}\left(\frac{p_n q_n}{\Sigma p_n q_n} + \frac{p_1 q_1}{\Sigma p_1 q_1}\right)$

$$W_{n+1\cdot 1} = \frac{1}{2}\left(\frac{p_n q_{n+1}}{\Sigma p_n q_{n+1}} + \frac{p_1 q_1}{\Sigma p_1 q_1}\right)$$

$$W_{n\cdot 1} - W_{n+1\cdot 1} = \frac{1}{2}\left(\frac{p_n q_n}{\Sigma p_n q_n} - \frac{p_n q_{n+1}}{\Sigma p_n q_{n+1}}\right)$$

If $p_{n+1}$ and $p_n$ are *approximately* equal and if for those relatives for which $p_n > p_1$, $\frac{p_n q_{n+1}}{\Sigma p_n q_{n+1}} > \frac{p_n q_n}{\Sigma p_n q_n}$ and for those relatives for which $p_n < p_1$, $\frac{p_n q_{n+1}}{\Sigma p_n q_{n+1}} < \frac{p_n q_n}{\Sigma p_n q_n}$ then $\frac{R_{n+1}}{R_n}$ will tend to be $< 1$ and the chain index will tend to diverge downward from the fixed base index. That is to say, in such a case the fixed base index will tend to move upward with respect to the chain index (which in this case moves horizontally) if the high relatives (*i.e.*, more than 100 per cent) are associated with larger quantities in period $(n+1)$ than in period $(n)$ and if the low relatives (*i.e.*, less than 100 per cent) are associated with smaller quantities in period $(n+1)$ than in period $(n)$. If, on the other hand, the high relatives are associated with smaller quantities in period $(n+1)$ than in period $(n)$ and the low relatives are associated with larger quantities in period $(n+1)$ than in period $(n)$ the chain index would tend to diverge upward with respect to the fixed base index.

In general, for any values for prices and quantities the chain index will tend to diverge downward with respect to the fixed base index, if

$$\frac{\Pi\left(\frac{p_n}{p_1}\right)^{W_{n\cdot 1}}}{\Pi\left(\frac{p_{n+1}}{p_1}\right)^{W_{n+1\cdot 1}}} < \Pi\left(\frac{p_n}{p_{n+1}}\right)^{W_{n+1\cdot n}}$$

The chain index will diverge upward from the fixed base index if the left-hand side of the above expression is greater than the right-hand side.

We may say for formula i as we did for formula v that if, in actual practice, important commodities rise in price more than the average (relative to a fixed base) and remain high in price in the succeeding period regardless of higher production and if important commodities fall in price more than the average (relative to a fixed base) and remain low in price regardless of smaller production (*i.e.*, if in general it is difficult to adjust production to prices and if more than one period is required for adjustment and in the meantime production is increased due to the stimulus of high prices *but not enough* to cause significant decline in price, and production of other commodities is curtailed on account of low prices *but not enough* to cause a significant rise in prices) then

$$\Pi\left(\frac{p_n}{p_1}\right)^{W_{n\cdot 1}}$$

will tend to be smaller, on account of $q_n$ in the weights, than

$$\Pi\left(\frac{p_{n+1}}{p_1}\right)^{W_{n+1\cdot 1}}$$

which will tend to be larger, on account of $q_{n+1}$ in the weights,

and

$$\Pi\left(\frac{p_{n+1}}{p_n}\right)^{W_{n+1\cdot n}}$$

will tend to equal 1 on account of the approximate equality of $p_{n+1}$ and $p_n$

and

$\frac{R_{n+1}}{R_n}$ will tend to $< 1$

and

the fixed base index will tend upward faster than the chain index or, in other words, the chain index will diverge downward.

If a situation leading to the divergence of chain and fixed base indexes, such as we have described, persists over several consecutive years the two indexes will move farther and farther apart. In fact the divergence will cumulate according to the law of compound interest.

Thus, suppose we have three years 1, 2 and 3 such that the chain index for the third year bears the ratio $R_3$ to the fixed base index ($P_{3\,1}$). That is to say let

$$R_3 = \frac{P_{2\,1} \cdot P_{3\,2}}{P_{3\,1}}$$

Suppose that the values of $p$ and $q$ of year 1 are repeated in year 4, those of year 2 are repeated in year 5, and those of year 3 are repeated in year 6. Then

$$R_6 = \frac{P_{2\,1} \cdot P_{3\,2} \cdot P_{4\,3} \cdot P_{5\,4} \cdot P_{6\,5}}{P_{6\,1}}$$

Since, by assumption, $P_{4\,3} = P_{1\,3}$, $P_{5\,4} = P_{2\,1}$, $P_{6\,5} = P_{3\,2}$, $P_{6\,1} = P_{3\,1}$, and $P_{1\,3} = \frac{1}{P_{3\,1}}$ we have

$$R_6 = \left(\frac{P_{2\,1} \cdot P_{3\,2}}{P_{3\,1}}\right)^2 = (R_3)^2$$

Therefore, if a situation resulting in the divergence of chain and base index is repeated in consecutive periods the two indexes will diverge cumulatively in geometric progression.

From the results presented above it is clear that the chain index not only deviates from the corresponding fixed base index, but the deviation is cumulative in accordance with the law of compound interest, and, in actual practice, the two indexes are apt to diverge more and more widely as the number of links in the chain increases.

We conclude that a chain index is an unreliable substitute for a fixed base index and should not be used unless the weights are constant, *i.e.*, unless the index satisfies the circular test and the chain and fixed base indexes coincide. This conclusion indicates another, *i.e.*, the best fixed base index to use for a comparison involving more than two periods is either the geometric average with fixed weights or the aggregative average with fixed weights.

## VI. SUMMARY

1. The object of the present study is that of measuring the average change of prices or quantities of two or more commodities during two or more periods. In other words, our problem is that of averaging relatives to secure index numbers.

2. In attacking this problem we confront at the outset the difficulties of (*a*) the selection of weights in cases for which, in actual practice, weights and relatives may or may not be correlated and (*b*) the selection of an average — from the familiar types — which will be adapted to the selected method of weighting.

3. Eleven formulas for index numbers were derived, for binary comparisons, in which

weights and averages were selected in such a way as to eliminate the bias between weights and the corresponding relatives. Nine of the eleven formulas satisfy the base reversal test.

4. Averages of relative prices, in order to satisfy the base reversal test in general, must have fixed weights when the base is reversed. That is to say, each relative must appear somewhere in the formula with the same weight when the index is figured forward and backward. The logical extension of the base reversal test for two periods is the circular test for three or more periods. Satisfaction of the circular test by an average necessitates the use of fixed weights.

5. *Exact* satisfaction of the *factor* reversal test is not of great practical importance in the construction of index numbers, and *exact* satisfaction of the *base* reversal test is not a necessary condition for the securing of a good index number if weights and averages are selected in such a way as to avoid weight correlation bias.

6. Chain indexes will, in general, differ from corresponding fixed base indexes for the same periods on account of differences in weighting alone. That is to say, even though the prices of all the articles remain fixed during two or more periods the chain index computed by any formula will, in general, deviate from the fixed base index computed according to the same formula; further, the source of the deviation is

the difference in weights of the chain and fixed base indexes.

7. Between any pair of consecutive periods the chain index will increase less rapidly (or decrease more rapidly) than the fixed base index computed by the same formula if a certain kind of relationship or correlation exists between the movements of prices and movements of weights for those periods. Likewise, the chain index will increase more rapidly (or decrease less rapidly) than the fixed base index if the converse of the relationship or correlation (indicated in the preceding statement) exists.

8. Specifically, if we use formula v, the "ideal," the "certain kind of relationship or correlation between the movements of prices and movements of weights" that will cause the chain index to increase less rapidly in any interval than the fixed base index is one in which the price relatives of period $(n)$ continue high in period $(n+1)$, but quantities increase and/or low price relatives of period $(n)$ continue low in period $(n+1)$, but quantities decrease so that the weights $p_1\, q_{n+1}$ are positively associated with high relatives and negatively associated with low relatives. In such cases $(p_1 q_n)$ would be less highly positively associated with high relatives and more highly negatively associated with low relatives.

9. The situation described in the preceding paragraph is apt to occur if the adjustment of the quantities to the stimulus of high prices or

the depressing effect of low prices are not in fact accomplished in the interval between the two periods in question. If the adjustments of quantities to prices take several periods the chain index would tend to diverge more and more widely downward from the fixed base index. This is more apt to be the situation for agricultural products than for the products of manufacturing and mining.

10. Even if there were no persisting correlation between relatives and weights of certain commodities, since deviations of the chain index from the fixed base index are cumulative "a run of luck" would at times in actual practice lead to a wider and wider divergence of the two indexes. For this reason alone the chain index is "suspect" and cannot be recommended as a reliable form of average.

11. In general, entirely aside from "runs of luck," the persistence of economic situations over a series of periods makes it likely that the chain index will, at times, diverge more and more widely (in one direction or the other) from the fixed base index.

12. When there is a significant divergence of the chain and fixed base indexes which is preferable? The answer depends upon the comparison we wish to make. If we desire merely to compare the average price of one period with that of the preceding period the link of the chain is the index to be preferred. If we desire to

compare the average price of each year $(n)$ and $(n+1)$, respectively, with the base year the fixed base index is to be preferred. If our object be that of *comparing each period* $(n)$ *and* $(n+1)$ *with the other and with the base simultaneously* neither the chain indexes nor the fixed base indexes (with the weights of two years only) is satisfactory.

13. The only way in which the fixed base indexes and the chain index can be made to give results which are not inconsistent with each other for an intercomparison of three or more periods is to use fixed or constant weights in constructing indexes of average prices for the periods to be compared. In other words, satisfaction of the "circular test" is indicated.

14. Therefore, for an intercomparison of average prices for three or more periods a weighted average with fixed weights is to be preferred. Such an average should satisfy the base reversal test (the geometric or aggregative average) and the fixed weights should depend upon the average prices and average quantities of all the periods in question. Since the weights have, in actual practice, less influence on the result than the relatives averaged, the method of averaging prices and of averaging quantities to secure weights is of decidedly less importance than the method of averaging the relatives. The aggregates (or arithmetic averages[1]) of prices and of

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[1] The kind of average selected in obtaining weights, being

quantities are, therefore, recommended as the factors to be used in securing fixed or constant weights.

15. The decision concerning the base to be used depends upon the particular comparison desired. If the object is an intercomparison of each index with every other one an average base period of all the constituent periods is preferred. Since constant weights are used the same result would be obtained by shifting the base to any desired year, by use of a factor, as that which would have been obtained by using the desired year in the first instance.

16. We may point out in this connection that for the base period the dispersion of price relatives is zero and that the choice of base determines the dispersion of price relatives for other periods.

17. The best index number of prices or quantities for a binary comparison is given by the "ideal" formula.[1] The aggregative average with arithmetic mean quantities or prices as weights[2] — depending on whether we are computing an index of prices or quantities — however, is a

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of much less significance on the result than the kind of average used in computing the price index, greater simplicity and ease of computation of one average as compared with others are dominating considerations. With adding machines available, arithmetic averages are about as easy to ascertain, for instance, as medians.

[1] Approved by Fisher, Pigou, Walsh, and Allyn Young.

[2] Approved by Edgeworth, Fisher, Marshall, and Walsh.

very close contender for first place and, on account of its greater simplicity and ease of computation, is generally to be preferred in actual practice.

18. The best index number of prices or quantities for an intercomparison of three or more periods is the aggregative average with fixed weights based upon arithmetic mean quantities of prices, respectively, for all the periods covered.

19. Geometric averages of relatives of prices or quantities with fixed value weights based on arithmetic averages of prices and quantities for the periods being compared, two or more, in actual practice for a large number of commodities for which the weights and relatives are not extremely dispersed, conform closely to the "ideal" and aggregative indexes described above. Problems may arise in which the geometric average of relatives is logically indicated and in such cases the geometric average of relatives is, of course, to be preferred even though the results differ substantially from other averages.

20. When we have adopted a certain base and constant quantities for an index which is being continued currently we should not think that our problem has been settled once and for all. The continuity of series of index numbers is necessarily interrupted by economic changes. New data may make both the base and the quantities obsolete. The data, therefore, should be constantly watched and the index should be

periodically revised to correspond to current available data.

In order to avoid correlation bias between weights and relatives in a time series for all possible cases of positive, zero, or negative correlation between relatives and weights it is absolutely necessary to use fixed or constant weights whether the index numbers are computed for two periods or for more than two periods. These fixed weights measure only approximately the relative importance of the different commodities in the different years. As new weights become available tests should be made of the effect of using such new weights. If the effect is significant these new fixed weights should, in general, supplant the old weights.

Comprehensive averages of price changes for widely different (from an economic point of view) places or periods of time may be impossible to obtain, not because of the lack of satisfactory methods of averaging, but because of the lack of homogeneous data to be averaged. In such cases we are limited to the comparison of price changes of those commodities bought and sold in economically significant and comparable amounts in the periods compared.

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